Limit Theorems for Jacobi Ensembles

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Jacobi Ensembles

A set of random variables is a (complex, $\beta = 2$) *Jacobi ensemble* if that can be described in the following equivalent ways: for parameters (n, m_1, m_2),

- $(\lambda_1, \ldots, \lambda_n) \in [0, 1]^n$ with joint pdf

$$C_{n,m_1,m_2} \cdot \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^2 \cdot \prod_{i=1}^{n} \lambda_i^{m_1 - n} (1 - \lambda_i)^{m_2 - n}.$$

- Eigenvalues of $n \times n$ matrix $X = AA^*/(AA^* + BB^*)$ where A, B are matrices of size $n \times m_1$ and $n \times m_2$ with i.i.d. standard (complex) Gaussian random variables. Equivalently, it is the squared generalized singular values of the pair A, B.

Jacobi Ensembles

A set of random variables is a (complex, $\beta = 2$) *Jacobi ensemble* if that can be described in the following equivalent ways: for parameters (n, m_1, m_2),

- Eigenvalues of a random matrix of the form $\pi\tilde{\pi}\pi$ where π and $\tilde{\pi}$ are independent $(m_1 + m_2) \times (m_1 + m_2)$ random orthogonal projections of ranks n and m_1, whose distributions are invariant under unitary conjugation.
- Eigenvalues of U^*U where U is the $m_1 \times n$ upper-left corner of an $(m_1 + m_2) \times (m_1 + m_2)$ Haar (unitary) matrix.
- Log-gas model, etc.

Jacobi Ensembles

Can generalize to any $\beta > 0$.

- CS values of certain bidiagonal matrix. (Edelman & Sutton 2003)
- Eigenvalues of certain tridiagonal matrix. (Killip & Nenciu 2004)

Jacobi Ensembles

Let

$$B_\beta = \begin{pmatrix} c_n s'_{n-1} & & & & \\ -s_{n-1}c'_{n-1} & c_{n-1}s'_{n-2} & & & \\ & -s_{n-2}c'_{n-2} & c_{n-2}s'_{n-3} & & \\ & & \ddots & \ddots & \\ & & & -s_1 c'_1 & c_1 \end{pmatrix}$$

with

$$\begin{aligned} c_i^2 &\sim \text{Beta}(\frac{\beta}{2}(m_1 - n + i), \frac{\beta}{2}(m_2 - n + i)) \\ {c'_j}^2 &\sim \text{Beta}(\frac{\beta}{2}j, \frac{\beta}{2}(m_1 + m_2 - 2n + 1 + j)), \end{aligned}$$

and $s_i = \sqrt{1 - c_i^2} = 1$, $s'_j = \sqrt{1 - {c'_j}^2}$.

Then, the eigenvalues of $B_\beta B_\beta^T$ has same distribution as Jacobi ensemble.

Limit theorems

If we sample n values from a certain distribution, then how would the "average" look like?

- Independent random variables: Law of large numbers.

$$\frac{X_1 + \ldots + X_n}{n} \to \mu$$

- Hermite ensembles (GOE, GUE, ...) : Wigner's semicircle law

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \to \frac{2}{\pi}\sqrt{1-x^2}dx$$

- Laguerre ensembles : Marchenko-Pastur law

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \to 1_{[\gamma_-,\gamma_+]}\frac{\sqrt{(x-\gamma_-)(\gamma_+-x)}}{2\pi\gamma x}dx$$

Limit theorems

How about Jacobi ensembles? Let parameters be (n, m_1, m_2).

1. If $m_1 + m_2 - 2n = o(n)$, then

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i} \to \frac{1}{\pi\sqrt{x(1-x)}}dx.$$

2. If $m_1/n \to p \geq 1$ and $m_2/n \to q \geq 1$ and $p + q > 2$, then

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i} \to \frac{p+q}{2\pi}\frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{x(1-x)}1_{[\lambda_-,\lambda_+]}dx,$$

where

$$\lambda_\pm = \left(\sqrt{\frac{p}{p+q}(1-\frac{1}{p+q})} \pm \sqrt{\frac{1}{p+q}(1-\frac{p}{p+q})}\right)^2.$$

Limit theorems

How about Jacobi ensembles? Let parameters be (n, m_1, m_2).

3. If $m_1 + m_2 - 2n = \omega(n)$ and if $\frac{m_1 - n}{m_1 + m_2 - 2n} \to \lambda$, then

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \to \delta_{\lambda}.$$

(Every convergence here is in probability.)

We can say more about the third case if we scale appropriately.

Limit theorems

Theorem

Assume that $\frac{n\beta}{2m_1} \to \gamma \in (0, 1]$, *and* $m_2 = \omega(n^2)$. *Then,*

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{\frac{m_2}{n}\lambda_i} \to c \cdot f_\gamma(cx)dx$$

weakly, where $c = 2\gamma/\beta$ *and* f_γ *is the density function of Marchenko-Pastur law with parameter* γ, *i.e.,*

$$f_\gamma(x) = \frac{\sqrt{(x - \gamma_-)(\gamma_+ - x)}}{2\pi\gamma x} \cdot 1_{[\gamma_+, \gamma_-]}$$

and $\gamma_\pm = (\sqrt{\gamma} \pm 1)^2$.

Fluctuations

One may think about the "deviation". For example,

- Random variables: Central limit theorem.
- β-Hermite ensemble: Arcsin law. For any polynomial ϕ,

$$\sum_{i=1}^{n} \phi(\lambda_i) = n \int_{-1}^{1} \phi(x) d\sigma(x) + \left(\frac{2}{\beta} - 1\right) \int_{-1}^{1} \phi(x) d\mu_H(x) + o(1/n),$$

  where σ has semicircle distribution, and

$$d\mu_H = \frac{1}{4}\delta_1 + \frac{1}{4}\delta_{-1} - \frac{dx}{2\pi\sqrt{1-x^2}}.$$

- β-Laguerre ensemble: Similar, we have

$$d\mu_L = \frac{1}{4}\delta_{\gamma_-} + \frac{1}{4}\delta_{\gamma_+} - \frac{dx}{2\pi\sqrt{(x-\gamma_-)(\gamma_+ - x)}}.$$

Fluctuations

In the case of "truly" Jacobi ensemble, we can calculate the deviation.

Theorem

If $m_1/n \to p$ and $m_2/n \to q$, then for any polynomial ϕ,

$$\sum_{i=1}^{n} \phi(\lambda_i) = n \int_{\lambda_-}^{\lambda_+} \phi(x) d\mu(x) + \left(\frac{2}{\beta} - 1\right) \int_{-1}^{1} \phi(x) d\mu_J(x) + o(1/n),$$

where μ is the limiting distribution, and

$$d\mu_J = \frac{1}{4}\delta_{\lambda_-} + \frac{1}{4}\delta_{\lambda_+} - \frac{dx}{2\pi\sqrt{(x-\lambda_-)(\lambda_+ - x)}}$$

References

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