LIMIT THEOREMS FOR JACOBI ENSEMBLES

CHIHEON KIM

1. Introduction

Let $\beta > 0$ be a constant and $n \geq 2$ be an integer. A beta-Jacobi ensemble (also called beta-MANOVA ensemble) is a set of random variables $(\lambda_1, \dots, \lambda_n) \in [0,1]^n$ with joint probability density function

$$(1) \qquad C_{\beta,m_1,m_2} \cdot \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta \cdot \prod_{i=1}^{n} \lambda_i^{\frac{\beta}{2}(m_1-n+1)-1}(1-\lambda_i)^{\frac{\beta}{2}(m_2-n+1)-1},$$

where n, m_1, m_2 are parameters, and C_{β,m_1,m_2} is a suitable normalizing constant. For $\beta = 1, 2$ and 4, the eigenvalues of $A^*A/(A^*A + B^*B)$ forms Jacobi ensembles where A, B are matrices of size $m_1 \times n$ and $m_2 \times n$ respectively, with independent standard real, complex or quarternion Gaussian entries.

In this classical setting, various kinds of model can be constructed for Jacobi ensembles. Starting with a random unitary matrix of size $(m_1 + m_2) \times (m_1 + m_2)$ taken from Haar distribution, the singular values of upper-left $m_1 \times n$ matrix are distributed according to 1 ([B05]). Tridiagonal model ([ES08], [KN04]) achieves the full generality that covers any $\beta > 0$ and removes the condition that m_1, m_2 are integers.

In this survey, we are mainly interested in the asymptotic behavior of empirical distribution $\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}$ of Jacobi ensembles as $n \to \infty$. It depends on the relative size of the parameters m_1 and m_2 to n. The most interesting case is that m_1 and m_2 grow linearly in n, where the limiting empirical distribution is distinguished from other known distributions. This was first observed by Watchter [W80] and rediscovered by Collins [B05], Dumitriu and Paquette [DP12]. In addition, Jiang [J12] observed that if m_1 is linear and m_2 is superlinear, then by scaling appropriately the limiting distribution becomes a scaled Marchenko-Pastur law.

2. Limit theorems

Let $X_1, \dots, X_n$ be independent, identically distributed random variables. By strong law of large number, its average converges to $\mathbb{E}X_1$ under some conditions on variance of the distribution of X_1. In similar sense, if "ensemble" of random variables $(\lambda_1, \dots, \lambda_n)$ is given (not necessarily independent), we can think about the limit of empirical distribution, $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$. For Hermite ensembles, it is known to be the famous Wigner's semicircle law. For Laguerre ensembles, the limiting distribution follows the Marchenko-Pastur law. Apart from two cases, the asymptotic behaviour of Jacobi ensembles is developed later; first observed by Watchter [W80].

Proposition 2.1 ([DP12])**.** *Let f be a continuous test function on $[0,1]$.*

(1) *If $m_1 + m_2 - 2n = o(n)$, then*
$$\frac{1}{n}\sum_{i=1}^n f(\lambda_i) \to_{\mathbb{P}} \frac{1}{\pi}\int_0^1 \frac{f(x)}{\sqrt{x(1-x)}}dx.$$
(2) *If $m_1/n \to p \geq 1$, $m_2/n \to q \geq 1$ and $p+q > 2$, then*
$$\frac{1}{n}\sum_{i=1}^n f(\lambda_i) \to_{\mathbb{P}} \int_0^1 f(x)d\mu(x),$$
where μ has density
$$d\mu(x) := \frac{p+q}{2\pi}\frac{\sqrt{(x-\lambda_-)(\lambda_+ - x)}}{x(1-x)}\mathbb{1}_{[\lambda_-,\lambda_+]}dx,$$
and
$$\lambda_\pm := \left[\sqrt{\frac{p}{p+q}(1-\frac{1}{p+q})} \pm \sqrt{\frac{1}{p+q}(1-\frac{p}{p+q})}\right]^2.$$
(3) *If $m_1 + m_2 - 2n = \omega(n)$ and if $(m_1 - n)/(m_1 + m_2 - 2n) \to \lambda$, then*
$$\frac{1}{n}\sum_{i=1}^n f(\lambda_i) \to_{\mathbb{P}} f(\lambda).$$

We are not going to give a rigorous proof here. The sublinear case is simple, since it is the limit of linear case where $p \to 1$ and $q \to 1$. For superlinear case, heuristically if $m_1 \gg n$ and $m_2 \gg n$, then $A^*A \approx \beta m_1 I$ and $B^*B \approx \beta m_2 I$. If m_1 and m_2 are in the same order, then the empirical distribution would converge to point mass at $m_1/(m_1+m_2)$. Or if $m_2 \gg n$ and $m_2 \gg m_1$, the heuristic predicts that
$$A^*A/(A^*A + B^*B) \approx A^*A/(\beta m_2),$$
hence it tends like Lagurre ensemble. More precisely,

Theorem 2.2 ([J12])**.** *Assume $n/m_1 \to \gamma \in (0,1]$ and if $m_2 = \omega(n^2)$, and let*
$$\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{\frac{\beta m_2}{2n}}.$$

Then, μ_n converges to μ in distribution, where μ has density $d\mu(x) = c \cdot f_\gamma(cx)dx$ with $c = 2\gamma/\beta$ and

$$f_\gamma(x) := \frac{\sqrt{(x-\gamma_-)(\gamma_+ - x)}}{2\pi\gamma x}\mathbb{1}_{[\gamma_-,\gamma_+]}$$

and $\gamma_\pm = (\sqrt{\gamma} \pm 1)^2$.

For the proof of linear cases, refer [W80] and [B05] for using s-transformation and free probability, and refer [DP12] for calculating moments via tridiagonal model and combinatorial arguments. Simulation using MATLAB is given in appendix B.

3. Fluctations

Once we figure out what the limiting distribution is, it would be interesting to find out how empirical distribution behaves in the second order. It is called *deviation* in [DE06]. For Hermite ensembles and Laguerre ensembles we get the following theorem.

Theorem 3.1 ([DE06], informal)**.** *Let ϕ be a polynomial. Then,*

(1) *For β-Hermite ensemble $(\lambda_1, \dots, \lambda_n)$,*
$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\phi(\lambda_i) = \int_{-1}^{1} \phi(x)d\sigma(x) + \frac{1}{n}\left(\frac{2}{\beta} - 1\right)\int_{-1}^{1}\phi(x)d\mu_H(x) + o\left(\frac{1}{n}\right),$$
where $d\sigma(x) = \frac{2}{\pi}\sqrt{1-x^2}dx$ (semicircle law), and
$$d\mu_H = \frac{1}{4}\delta_1 + \frac{1}{4}\delta_{-1} - \frac{dx}{2\pi\sqrt{1-x^2}}.$$

(2) *For β-Laguerre ensemble $(\lambda_1, \dots, \lambda_n)$ with parameter m s.t. $n\beta/(2m) \to \gamma \leq 1$,*
$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\phi(\lambda_i) = \int_{-1}^{1} \phi(x)d\nu(x) + \frac{1}{n}\left(\frac{2}{\beta} - 1\right)\int_{-1}^{1}\phi(x)d\mu_L(x) + o\left(\frac{1}{n}\right),$$
where $d\nu(x) = \frac{1}{2\pi\gamma}\sqrt{(\gamma_+ - x)(x-\gamma_-)}dx$ (Marchenko-Pastur law), $\gamma_\pm = (\sqrt{\gamma} \pm 1)^2$, and
$$d\mu_L = \frac{1}{4}\delta_{\gamma_+} + \frac{1}{4}\delta_{\gamma_-} - \frac{dx}{2\pi\sqrt{(\gamma_+ - x)(x-\gamma_-)}}.$$

In [DP12], the analogous result for Jacobi ensemble is presented.

Theorem 3.2. *Let $(\lambda_1, \dots, \lambda_n)$ be a β-Jacobi ensemble with parameters (n, m_1, m_2). Assume $m_1/n \to p \geq 1$, $m_2/n \to q \geq 1$ and $p + q > 2$. For any polynomial ϕ,*

$$\sum_{i=1}^{n} \mathbb{E}\phi(\lambda_i) = n \int_{\lambda_-}^{\lambda_+} \phi(x) d\mu(x) + \left(\frac{2}{\beta} - 1\right) \int_{\lambda_-}^{\lambda_+} \phi(x) d\mu_J(x) + o\left(\frac{1}{n}\right)$$

where μ is as defined in theorem 2.1, and μ_J is the signed measure with density

$$d\mu_J = \frac{1}{4}\delta_{\lambda_+} + \frac{1}{4}\delta_{\lambda_-} - \frac{dx}{2\pi\sqrt{(\lambda_+ - x)(x - \lambda_-)}} \mathbb{1}_{(\lambda_-, \lambda_+)}.$$

References

[W80] K. W. Wachter, The limiting empirical measure of multiple discriminant ratios, Ann. Statist. Volume 8, Number **5** (1980), 937-957.

[KN04] R. Killip and I. Nenciu, Matrix models for circular ensembles, Int. Math. Res. Not. **50** (2004), 26652701.

[B05] B. Collins, Product of random projections, Jacobi ensembles and universality problems arising from free probability, Probab. Theory Relat. Fields **133** (2005), 315-344.

[DE06] I. Dumitriu and A. Edelman, Global spectrum fluctuations for the β-Hermite and β-Laguerre ensembles via matrix models, J. Math. Phys. **47** (2006), no. 6, 063302, 36.

[ES08] A. Edelman and B. D. Sutton, The beta-Jacobi matrix model, the CS decomposition, and generalized singular value problems, Found. Comput. Math. **8** (2008), no. 2, 259285.

[DP12] I. Dumitriu and E. Paquette, Global fluctuations for linear statistics of β-Jacobi ensemble. Random Matrices: Theory Appl. **01** (2012), 1250013.

[J12] T. Jiang, Limit theorems on beta-Jacobi ensembles. To appear in Bernoulli (2012).

Appendix A. MATLAB codes for simulations

We used the codes from [ES08] to generate Jacobi ensembles.

Code 1. Simulating the limiting distribution of Jacobi ensemble when $m_1 + m_2 - 2n = o(n)$.

```
function jacobi_sublinear(beta, trial, list)
% Case of m_1+m_2-2n = o(n);
% Especially m_1=n+log(n), m_2=n+log(n).
% Empirical distribution obeys the arcsin law asymptotically
t=trial;
e=[];
syms x f
f=1/(pi*sqrt(x*(1-x)));
F = real(double(subs(f,x,[.01:.02:.99]))); F(F==inf)=0;
M = max(F)*1.2;
for j = 1:length(list)
    n = list(j);
```

```
    e=[];
    for i = 1:t
        g = jacobiensemble(beta,n,floor(log(n)),floor(2*log(n)));
        e = [e;g];
    end
    [h,hn,xspan]=histn(e,0,.02,1);
    axis([-0.1, 1.1, 0, M]); hold on
    plot([.01:.02:.99],F,'red','LineWidth',2); drawnow;
    hold off
end
```

Code 2. Simulating the limiting distribution of Jacobi ensemble when $m_1/n \to p$ and $m_2/n \to q$.

```
function jacobi_linear(beta, par, trial, list)
% Case of m_1/n = a, m_2/n = b
% Empirical distribution obeys the law f asymptotically
t=trial;
a=par(1);
b=par(2);
A=a/(a+b)*(1-1/(a+b));
B=1/(a+b)*(1-a/(a+b));
r=(sqrt(A)-sqrt(B))^2;
s=(sqrt(A)+sqrt(B))^2;
e=[];
syms x f
f=(a+b)*sqrt((x-r)*(s-x))/(2*pi*x*(1-x));
F = real(double(subs(f,x,[.01:.02:.99]))); F(F==inf)=0;
M = max(F)*1.2;
for j = 1:length(list)
    n=list(j);
    e=[];
    for i = 1:t
        g = jacobiensemble(beta,n,a*n-n,b*n-n);
        e = [e;g];
    end
    [h,hn,xspan]=histn(e,0,.02,1);
    axis([0, 1, 0, M]); hold on
    plot([.01:.02:.99],F,'red','LineWidth',2); drawnow;
    hold off
end
```

Code 3. Simulating the limiting distribution of scaled Jacobi ensemble when $m_2 = o(n^2)$ and $n/m_1 \to \gamma$.

```
function jacobi_superlinear(beta, a, trial, list)
% Case of m_1/n = a, m_2 = n^3
% (beta*m_2/2n)*lambda asymptotically obeys Marchenko-Pastur law
t=trial;
gamma=1/(a);
c=2*gamma/beta;
r = (sqrt(gamma)-1)^2/c;
s = (sqrt(gamma)+1)^2/c;
e=[];
syms x f
f=c*sqrt((x-s)*(r-x))/(2*gamma*pi*x);
xspan=[r:(s-r)/50:s];
F = double(subs(f,x,xspan)); F(F==inf)=0;
M = 1.5*max(F);
for j = 1:length(list)
    n = list(j);
    e=[];
    for i = 1:t
        g = jacobiensemble(beta, n, a*n-n, floor(n^3)-n)*n^2*beta/2;
        e = [e;g];
    end
    histn(e,r,(s-r)/50,s);
    axis([r-1, s+1, 0, M]); hold on
    plot(xspan,F,'red','LineWidth',2);
    drawnow;
    hold off
end
```

Code 4. Generating normalized histogram by Edelman (`http://web.mit.edu/18.338/www/handouts/handout2.pdf`).

```
function [h,hn,xspan]=histn(data,x0,binsize,xf);
%HISTN Normalized Histogram.
% [H,HN,XSPAN] = HISTN(DATA,X0,BINSIZE,XF) generates the normalized
% histogram of area 1 from the values in DATA which are binned into
% equally spaced containers that span the region from X0 to XF
% with a bin width specified by BINSIZE.
%
% X0, BINSIZE and XF are all scalars while DATA is a vector.
% H, HN and XSPAN are equally sized vectors.
```

```
%
% References:
% [1] Alan Edelman, Handout 2: Histogramming,
% Fall 2004, Course Notes 18.338.
% [2] Alan Edelman, Random Matrix Eigenvalues.
%
% Alan Edelman and Raj Rao, Sept. 2004.
% $Revision: 1.1 $ $Date: 2004/09/10 17:11:18 $
xspan=[x0:binsize:xf];
h=hist(data,xspan); % Generate histogram
hn=h/(length(data)*binsize); % Normalize histogram to have area 1
bar(xspan,hn); % Plot histogram
```

Appendix B. Simulations

Figure 1, 2, 3 represent results from simulating empirical distribution of Jacobi ensembles for the case $n = 100$ with 100 trials. Figure 4, 5, 6 support that theorem 2.2 would hold for $m_2 = \Omega(n^{1+\epsilon})$.

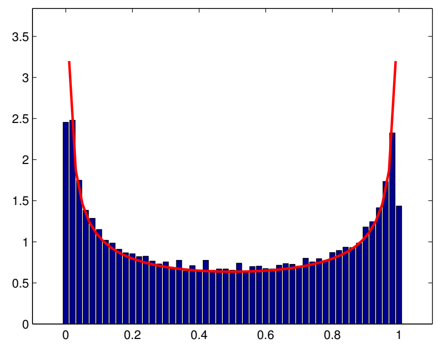

Figure 1. $\beta = 2$, $m_1 = n + \log n$, $m_2 = n + 2\log n$

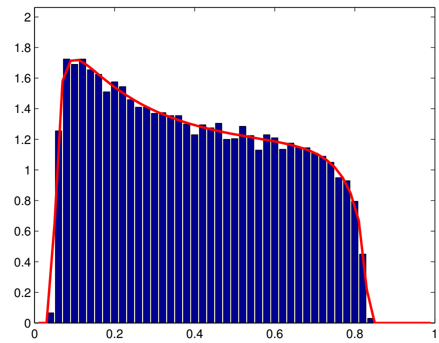

FIGURE 2. $\beta = 2$, $m_1 = 2n$, $m_2 = 3n$

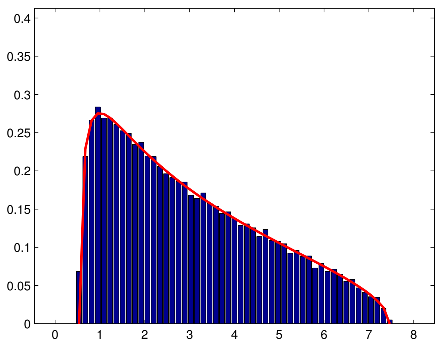

FIGURE 3. $\beta = 2$, $m_1 = 3n$, $m_2 = n^3$ (Scaled with factor n^2)

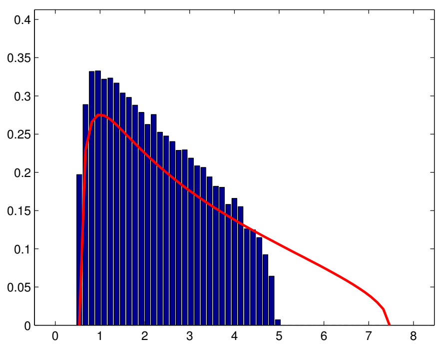

FIGURE 4. $\beta = 2$, $m_1 = 3n$, $m_2 = n^{1.5}$, $n = 100$

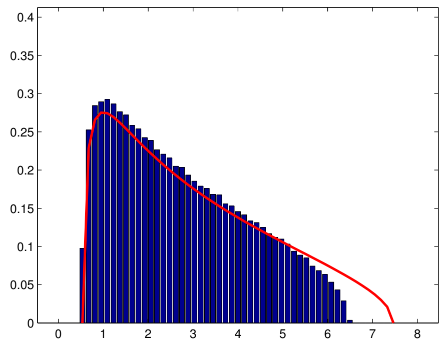

FIGURE 5. $\beta = 2$, $m_1 = 3n$, $m_2 = n^{1.5}$, $n = 1000$

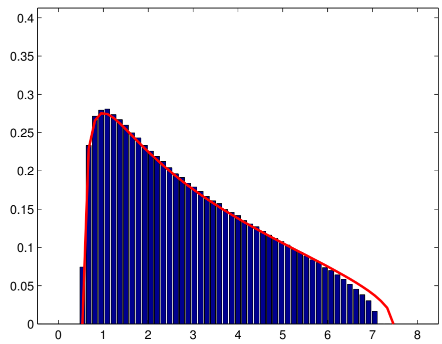

FIGURE 6. $\beta = 2$, $m_1 = 3n$, $m_2 = n^{1.5}$, $n = 10000$