Asymptotics of Random Lozenge Tilings

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- Lozenge tilings and symmetric polynomials.
- Eigenvalues of finite dimensional GUE matrices.

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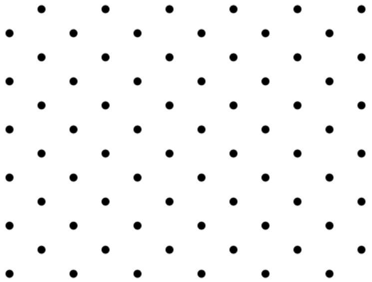

Figure : A triangular lattice from Wikipedia

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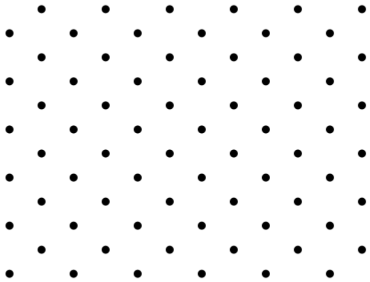

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- And draw a domain Ω on it

What is a Lozenge Tiling?

- *Lozenge tiling*: tile the domain Ω by rhombi of three types.

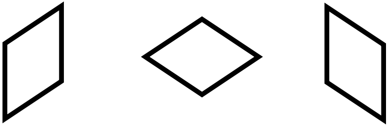
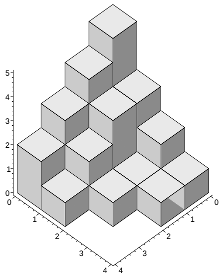


Figure : The lozenges. Figure from [VGGP].

Surfaces and Lozenge Tilings

- Stack unit cubes in the plane.



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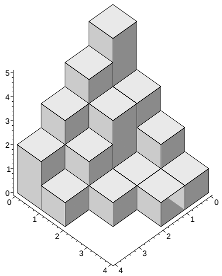

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- And look straight down (in the $(1, 1, 1)$ direction).

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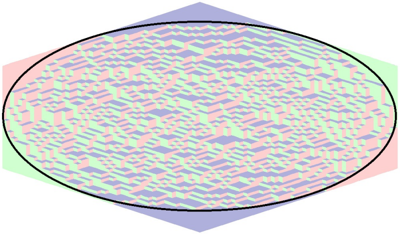

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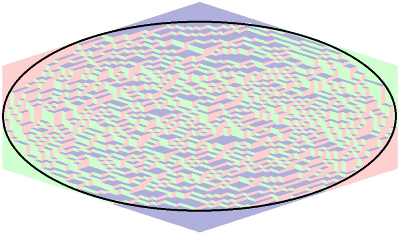

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Result: Left-most horizontal lozenges are distributed according to GUE eigenvalues

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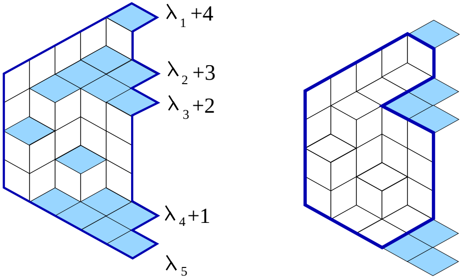

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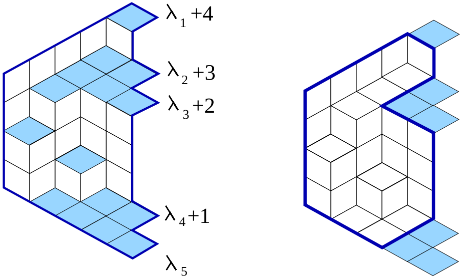

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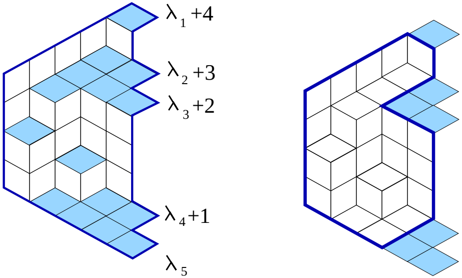

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- $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a *signature* when $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.
- The signature λ encodes the domain Ω_λ.

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- A symmetric polynomial is a polynomial unchanged by permuting variables.

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- Since they form a basis, we can define an inner product where they form an orthonormal basis

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

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  for all partitions ν.
- We will also need normalized Schur functions:

$$S_\lambda(x_1, \dots, x_k; N, 1) = \frac{s_\lambda(x_1, \dots, x_k, \overbrace{1, \dots, 1}^{N-k})}{s_\lambda(\underbrace{1, \dots, 1}_{N})}$$

Asymptotics

- As it turns out the distribution of Υ_λ^k is given by:

$$\mathbb{P}\left(\Upsilon_\lambda^k = \eta\right) = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

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  where $1^k = (1, \ldots, 1)$.
- Asymptotics of Schur polynomials $\implies$ Asymptotics of Υ_λ^k.

Theorem ([VGGP])

Let $\lambda(N)$, $N = 1, 2, \dots$ be a sequence of signatures. Suppose that there exist a non-constant piecewise-differentiable weakly decreasing function $f(t)$ such that

$$\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f\left(\frac{i}{N}\right) \right| = o(\sqrt{N}),$$

as $N \to \infty$ and also $\sup_{i,N} |\lambda_i(N)/N| < \infty$. Then for every k as $N \to \infty$ we have

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$

in the sense of weak convergence

Comments

$$E(f) = \int_0^1 f(t)\,dt, \quad S(f) = \int_0^1 f(t)^2\,dt - E(f)^2 + \int_0^1 f(t)(1-2t)\,dt.$$

$S(f)$ is always positive when we consider weakly decreasing functions $f(t)$. f encodes geometric information about the turning point and the curvature of the limit shape at that point.

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- $\mathbb{E} B_k(x;\Upsilon^k_\lambda + \delta_k)$ is the moment generating function of Υ^k_λ.
- $\delta_k = (k-1, k-2, \dots, 0)$, for $k=1$ we see $\mathbb{E} B_k(x;\Upsilon^k_\lambda + \delta_k) = \mathbb{E}\exp(x\Upsilon^1_\lambda)$

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- All we want to show then is that

$$\mathbb{E}B_k(x;\Upsilon^k_{\lambda(N)} + \delta_k) \to \mathbb{E}B_k(x;\mathbb{GUE}_k),$$

for all x in a neighborhood of $(0,\ldots,0)$.

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