Dyson Brownian Motion

① 2/20/13

Sym Eig Pert Theory

Differentiate $Aq = \lambda q$

$A' = E$
with $A'' = 0$ for simplicity
A is a "line" of Matrices
$A(t) = A + Et$

$q^T q' = 0 = q^T q'' = \ldots$

der	
1	$Eq + Aq' = \lambda' q + \lambda q'$
2	$2Eq' + Aq'' = \lambda'' q + 2\lambda' q' + \lambda q''$
3	$3Eq'' + Aq''' = \lambda''' q + 3\lambda'' q' + 3\lambda' q'' + \lambda''' q$

$$q^T E q^{(k-1)} = \lambda^{(k)}$$

$$\lambda' = q^T E q$$
$$\lambda'' = q^T E q^{(1)}$$

$$Eq + Aq^{(1)} = qq^T Eq + \lambda q^{(1)}$$

$$(A - \lambda I)q^{(1)} = -(I - qq^T)Eq \quad \text{An equation in } q^\perp$$

$$q^{(1)} = -\cancel{2}(A - \lambda I)^{-1} Eq$$

$$\lambda'' = -2q^T E(A - \lambda I)^{-1} Eq$$

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without loss of generality assume
$A = \text{diagonal} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$\lambda_i' = E_{ii}$$

$$\lambda_i'' = -2\sum_{j \neq i} \frac{|E_{ij}|^2}{\lambda_j - \lambda_i}$$

Add a little GOE matrix

$$c \cdot \sqrt{h} \cdot \begin{bmatrix} a & a/\sqrt{2} & \cdots \\ a/\sqrt{2} & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

Dyson Brownian Motion

$$d\lambda_i = c \cdot \sqrt{h}(G)$$

$$c\,dW - \frac{2hc^2}{2} \sum_{j \neq i} \frac{G^2}{\lambda_j - \lambda_i}$$

on average this is 1

$$d\lambda_i = c\,dW_i - dt \cdot \frac{c^2}{2} \cdot \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i}$$

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Eigenvalues as a change of variables

$$S = \begin{bmatrix} p & s \\ s & r \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos r \end{bmatrix}$$

$$p, r, s \longleftrightarrow \lambda_1, \lambda_2, \theta$$

$$\text{Jacobian Matrix} = \begin{vmatrix} dp/d\theta & dp/d\lambda_1 & dp/d\lambda_2 \\ dr/d\theta & dr/d\lambda_1 & dr/d\lambda_2 \\ ds/d\theta & ds/d\lambda_1 & ds/d\lambda_2 \end{vmatrix}$$

$$p = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$$
$$r = \lambda_2 \cos^2\theta + \lambda_1 \sin^2\theta$$

ectc

$$\det J = \lambda_1 - \lambda_2$$

$$\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(p^2+r^2+2s^2)} \, dp\, dr\, ds = \frac{1}{\sqrt{2}\pi^{3/2}} |\lambda_1 - \lambda_2| e^{-\frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2^2} \, d\lambda_1 \, d\lambda_2 \, d\theta$$

can integrate out $d\theta$

$$\frac{1}{\sqrt{2}\pi^{1/2}} |\lambda_1 - \lambda_2| e^{-\frac{1}{2}\lambda_1^2 + \lambda_2^2}$$

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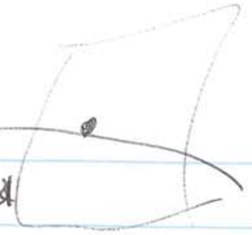
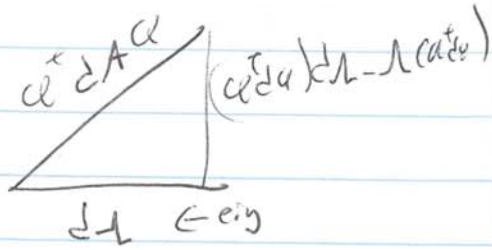
$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$dQ = (Q^T dQ)_{2\times}$



$$dQ = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} d\theta$$

$$Q^T dQ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d\theta$$



$$Q^T Q = I$$

$$dQ^T Q + Q^T dQ = 0$$
$$Q^T dQ = -(Q^T dQ)^T$$

Wielandt-Hoffman

$\|dA\|_F \geq \|d\Lambda\|_F$

$$A = Q \Lambda Q^T$$
$$dA = Q\, d\Lambda\, Q^T + dQ\, \Lambda\, Q^T + Q \Lambda\, dQ^T$$

$$Q^T dA\, Q = d\Lambda + Q^T dQ\, \Lambda - \Lambda\, Q^T dQ$$

diag $= d\lambda_i$

off-diag $= q_i^T dq_j (\lambda_j - \lambda_i)$

$$\prod_{i<j} |\lambda_i - \lambda_j|$$

$$\frac{\partial (Q^T dA\, Q)}{\partial \lambda_i} = 1 \qquad \frac{\partial (Q^T dA\, Q)}{\partial (Q^T dQ)_{ij}} = \lambda_j - \lambda_i$$

$i<j$