① Lecture 2 Feb 11, 2013

From Gaussian Ensembles to Stochastic Operators

1. (Real) GOE "Gaussian orthogonal Ensemble"

   A=randn(n);

   S=(A+A')/$\sqrt{2n}$

Fact: $\lambda_{max}(S) \approx 2$ $\lambda_{min}(S) \approx -2$ when $n \gg 1$

Fact: Normalized Histogram $\approx \frac{1}{2\pi}\sqrt{4-x^2}$

(A+A')/2 has $N(0,1)$ on diagonal + $N(0,\frac{1}{2})$ off

2. (Complex) GUE "Gaussian Unitary Ensemble"

   A=randn(n)+i*randn(n);

   S=(A+A')/(2$\sqrt{n}$)

Two Facts above remain true.

Notes: p. 48, 135-136 for above.

p 63 for below

3. Chi-distribution and orthogonal invariance

Let v_n = randn(n,1), i.e. an n-vector with independent standard normal entries

a) $\|v_n\|$ is known as a χ_n variable

its prob density is $\dfrac{x^{n-1}e^{-x^2/2}}{2^{n/2-1}\Gamma(n/2)}$ on $\mathbb{R}_+$

(Obviously $\chi^2_{n_1} + \chi^2_{n_2} = \chi^2_{n_1+n_2}$)
Degrees of freedom need not be integers, however

b) Orthogonal Invariance

If Q is fixed or random + independent of v_n, then Qv_n and v_n are identically distributed.

Joint density $= (2\pi)^{-\frac{n}{2}} e^{-\|v_n\|^2/2}$
which only depends on $\|v_n\|$

$$V_n = \|v_n\| \cdot \frac{v_n}{\|v_n\|}$$

χ_n uniform on sphere

independent

4. Central Limit Theorem

Scalar: $\frac{1}{\sqrt{K}} \sum_{i=1}^{K} x_i \to$ Standard Normal
if x_i have mean 0
variance 1 (+ bounded moments)

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Suppose X_i is an $n \times n$ matrix with elements of mean 0 + variance 1 (+ finite moments)

$$X = \frac{1}{\sqrt{K}} \sum_{i=1}^{K} X_i$$

Then $(X + X')/\sqrt{2n}$ converges to a GOE. Similary for a GUE

Hence, "universality" already seems likely.

5 Universality is a physics originating term somewhat related to what appears when considering statistical ensembles. It is an indicator of a central limit theorem effect. If a non-Gaussian random matrix acts approximately, asymptotically, or on average like a Gaussian matrix, we say that universality is occuring.

6. The "O" in GOE, the "U" in GUE

Let $A = \text{randn}(n)$

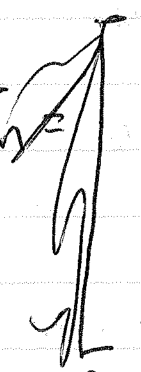
Clearly $Q_1 A Q_2$ is also $\text{randn}(n)$ for orthogonal Q_1, Q_2

So is QAQ' (rows + columns rotated!)

Thus $(QAQ') + (QAQ')^T$ is distributed as $A+A'$

Same with GUE for unitary matrices

7. <u>Theorem</u>: Let T_n be the random sym tridiagonal Matrix

$T_n =$ 

with diagonal: $\sqrt{2}\cdot\text{randn}(n,1)$

off-diagonal: $[\chi_{\beta(n-1)}, \chi_{\beta(n-2)}, \ldots, \chi_1]$

Divide by $\sqrt{n\beta}$ to get eigenvalues approx in $(-2,2)$

When $\beta=1$ the eigenvalues of T_n are distributed the same as the GOE

When $\beta=2$ they are the same as GUE

$\beta=4$ " " " " " GSE

All $\beta>0$ make sense

<u>Comment</u>: To compute all eigenvalues of Gaussian ensembles efficiently one could use MATLAB's dense eig (calls LAPACK) or sparse eig (calls ARPACK) but both are inefficient.

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Let $X = \text{randn}(n)$

$S = \frac{X + X'}{\sqrt{2}}$

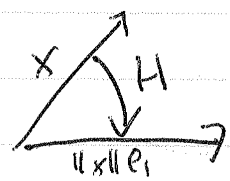
diagonal: $\sqrt{2}G$ i.e. variance $= 2$

off-diag: variance $= \frac{2}{2} = 1$

Householder Reflector from numerical computation

Given x construct H orthogonal so that

$$Hx = \|x\| e_1 \qquad \left(e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right)$$



Details don't matter.

One construction $v = x - \|x\| e_1$ external angle bisector

$H = I - \frac{2vv^T}{v^T v}$

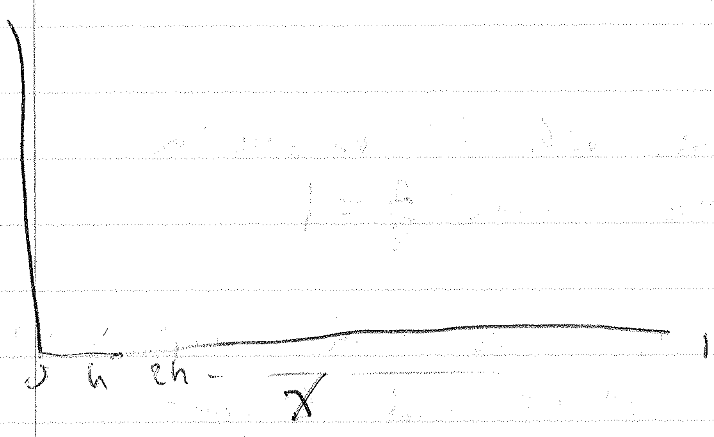
On computers other choices maybe used for stability

H is thought of as a tool to insert zeros

Usual Tridiagonal

$$\begin{pmatrix} G\sqrt{2} & \chi_{n-1} & \\ \chi_{n-1} & \boxed{\quad GOE_{n-1} \quad} \\ \vdots & \\ \vdots & \end{pmatrix}$$

Brownian Motion + White Noise

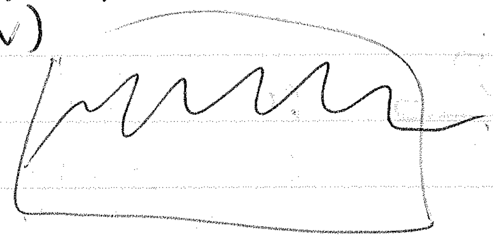


$h = 1/n$

$x = [0:h:1]$

$dW = \text{randn}(n+1,1) * \text{sqrt}(h)$

$W = \text{cumsum}(dW)$

$\text{plot}(x,W) =$ 

Note that $W(x)$ is normal with mean 0 & variance $h+h+\dots+h = x$

so $W(x)$ makes sense as a random variable

$$dW = (\text{standard normal}) \cdot \sqrt{dx}$$

$W'(x) = \frac{dW}{dx} \approx \frac{dW}{h}$ discrete time noise process

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Operators on $[0,\infty)$

Discrete $[f(0), f(h), f(2h), \ldots]$

Continuous $f(x)$

1st Derivative: Discrete: $\frac{1}{2h}\begin{bmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{bmatrix}\begin{bmatrix} f(0) \\ f(h) \\ \vdots \\ \vdots \end{bmatrix} \ldots$

Continuous: $f(x) \to \frac{d}{dx} f(x)$

2nd Derivative: $\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$

Multiplication by x $h\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & \ddots \end{bmatrix}$ $f(x) \to x f(x)$

Multiplication by $W'(x)$ $\frac{1}{\sqrt{h}}\begin{bmatrix} G & & \\ & \ddots & \\ & & G \end{bmatrix}$ $\frac{1}{\sqrt{h}} \cdot \text{diag}(\text{randn}(N,1))$

Let T = discretization of $\frac{d^2}{dx^2} - x + \frac{2}{\sqrt{\beta}} W'(x)$

then $\text{diag}(T) = [-\frac{2}{h^2} + h \cdot i + \text{randn}()/\text{sqrt}(h); \text{ for } i=1:N]$

$\text{supd}(T) = -2/h^2 * \text{ones}(N-1,1)$

Easy Theorem: The upper left of the tridiagonal model

$\chi_n \tilde{\approx} \sqrt{n} + \frac{G}{\sqrt{2}}$ converges