3/18/2013

(1)

Hermite ($\beta=2$) Ensemble spacings

Let $k(x,y) = \sum_{i=0}^{n-1} \phi_i(x)\phi_i(y)$

$\phi_i(x) = \pi_i(x)\sqrt{w(x)}$

$$\pi_i(x) = H_i(x)\left(\sqrt{\pi}\, i!\, 2^i\right)^{-1/2}$$
$$w(x) = e^{-x^2}$$

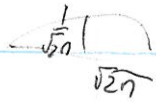
"Hermite_kernel(x)" computes $k_{ij} = k(x_i, x_j)$ for i,j

Recall $\frac{1}{n}k(x,x)$ computes the level density for the GUE with this normalization

A=randn(n) + i randn(n)

$S = \frac{A+A'}{\sqrt{8}}$

limiting semicircle scaling has $r = \sqrt{2n}$



i.e. exact density $= \frac{1}{n}k(x,x)$

~~limiting density~~

asymptotic density $= \frac{2}{\pi\sqrt{2n}}\sqrt{2n - t^2}$

Normalize to $[-2, 2]$ $\qquad \xi = \cancel{x} \cdot \sqrt{\frac{2}{n}}$ $\qquad \xi$ constant

$$\frac{1}{n} K(x, x)\, dx$$

$$= \frac{1}{n} K\left(\xi\sqrt{\frac{n}{2}}, \xi\sqrt{\frac{n}{2}}\right)\sqrt{\frac{n}{2}}\, d\xi$$

$$= \frac{1}{\sqrt{2n}} K\left(\xi\sqrt{\frac{n}{2}}, \xi\sqrt{\frac{n}{2}}\right)$$

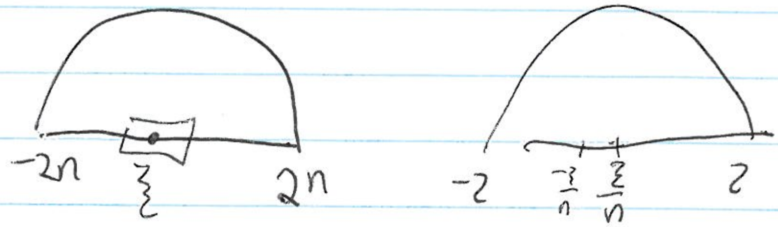
Normalize to $[-1, 1]$ $\qquad \xi = x \frac{1}{\sqrt{2n}}$

$$\sqrt{\frac{2}{n}} K\left(\xi\sqrt{2n}, \xi\sqrt{2n}\right)$$

Normalize to

$r = [-2n, 2n]$ $\qquad x = \frac{\xi}{\sqrt{2n}}$ $\qquad \xi$ constant

$$\frac{1}{n\sqrt{2n}} K\left(\frac{\xi}{\sqrt{2n}}, \frac{\xi}{\sqrt{2n}}\right) \to 1 \text{ for all } \xi.$$



uniform

③

Take $\quad \xi \in \left[\frac{-c}{n}, \frac{c}{n}\right]$

approx # of eigenvalues: $\cancel{\ }\ \frac{4c}{\pi}$

~~could take~~ $\cancel{c^n} \times \cancel{100}$ ~~or~~ ~~even~~ $\sqrt{n}$

~~Taken~~ Practice: ok to discard top 25% + bottom 25%

All these eigenvalues will appear uniformly distributed

<u>Take two eigenvalues</u>

Density is $\frac{1}{\binom{n}{2}} K\begin{pmatrix} x & y \\ x & y \end{pmatrix}$ converges to $\cancel{\frac{\sin(x-y)}{(x-y)}}$ $\left[\frac{\sin \ldots}{x-\ldots}\right.$

~~normalized to 1 as $x \to y$~~

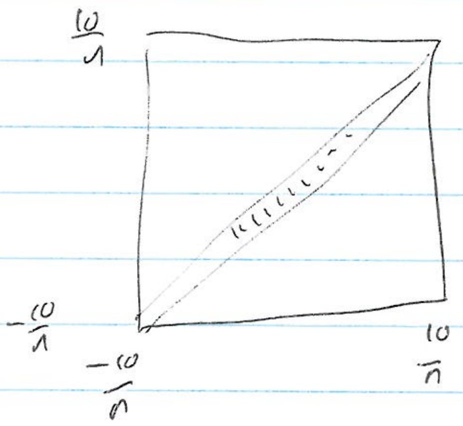
or $\quad 1 - \left[\frac{\sin(\xi - \eta)}{(\xi - \eta)}\right]^2$

This is more or less 1 i.e. on

$\left[-\frac{c}{n}, \frac{c}{n}\right] \times \left[\frac{-c}{n}, \frac{c}{n}\right]$ for large enough c

the eigenvalues behave as if they are uniformly dis[tributed]

$$1 - \left(\frac{\sin \delta}{\delta}\right)^2 : \quad \begin{array}{ll} 1 & 0.29 \\ 5 & 0.96 \\ 10 & 0.997 \end{array}$$



If level spacing is $\rho(t)$ in dt
there are $\rho(t)\cdot dt$ eigenvalues/trial
so avg spacing is $\frac{\text{length}}{\#\text{eigs}} = \frac{dt}{\rho(t)\cdot dt} = \frac{1}{\rho(t)}$

Multiply by $\rho(t)$ makes average spacing uniform

$\cancel{\rho(t) = \frac{1}{\beta\sqrt{2\pi}}\sqrt{2\pi - t^2}}$

$$d\delta\left(1 - \frac{\sin^2\delta}{\delta^2}\right) = \#\text{ with spacing } \delta$$

$$\frac{\{\#(i,j) \text{ with spacing } \delta\}}{\#(i,j)}$$

$$-\frac{\partial^2}{\partial a \partial b} E(a,b) = \text{Prob eig } (a) \text{ and at } (b)$$
but not in between

Condition on one eig at (a))

i.e. $\dfrac{-\frac{\partial^2}{\partial a \partial b} E(a,b)}{\text{level density at } (a)}$ & you get

Prob eig at a and next at b.

$I - kX$

$$f(t) \to \int_{-t}^{t} K(t,u)\, f(u)\, du$$

with uniform spacing and N samples

$$\frac{1}{N}\left\{\#\text{ pairs } (j_1, j_2) : 1 \le j_1, j_2 \le N, \ \delta \in [a,b]\right\}$$

$$\approx \int_a^b \left(1 - \left(\frac{\sin 2\pi u}{2\pi u}\right)^2\right) du$$

$$\cancel{\frac{1}{N}\left\{\#j : \ \theta_j \in (a,b)\right\} = b-a}$$

$$E(\{a,b\}) = P \text{ no eigenvalues in } [a,b]$$

For us $\det(1 - K\chi_{[a,b]})$

$$= \det\left(I - \left[\int_a^b \left(\sum \phi_i(x)\phi_j(x)\right) dx\right]_{0 \le i,j \le k-1}\right)$$

(1) $E(a,b) - E(a, b+db)$

$= \text{Prob} \ \underset{a}{[} \ \text{no} \ \underset{b}{]} \ \text{maybe} \ \underset{b+db}{]}$

$= P(\text{eig around } db$ but not in $(a,b))$

$- \underset{a}{[} \ \text{no} \ ] \ \text{no} \ \underset{b+db}{]}$

(2) $E(a+da, b) - E(a+da, b+db)$

maybe no yes $=$ Prob eig

$\text{Prob}(\text{(1)} - (1))$ no no x

Equilibrium Measure

$$T = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}$$

$\frac{1}{n} \sum_{i=1}^{n} \delta(x-\lambda_i)$ = ~~empirical~~ weight function

e.g. converge to normal

What is $\frac{1}{n} \sum \delta(x-\lambda_i)$?

Equilibrium Measure

Converge to a semi-circle

Assume $w(x) = e^{-V(x)}$ (canonical example is x^2

$$\text{PV}\int \frac{\psi(y)}{x-y}\, dy = \frac{1}{2\pi} V'(x)$$