North Pole Problem in Random Orthogonal Matrices

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- Mx_0 is uniformly distributed on the unit sphere. (well known)
- Without loss of generality, we fix x_0 at the "North Pole",

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

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- What is the probability density function for the random variable $x_0' M^k x_0$ in any n-dimensional space?

Numerical Results

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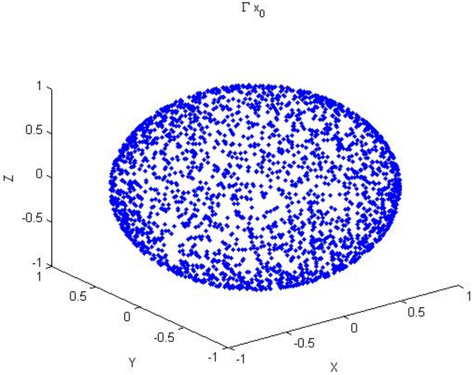
- The direction of each column vector of M is again randomized by multiplying 1 or -1 to avoid bias in MATLAB.
- The e_1 component (or say x component) of $M^k x_0$ is $x_0' M^k x_0$.

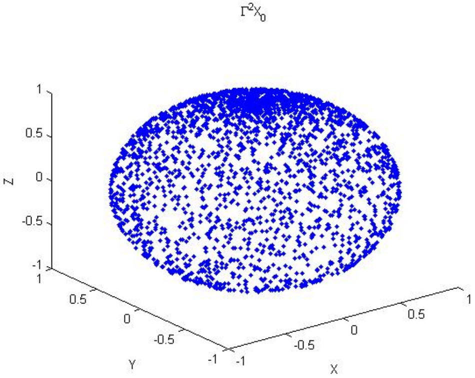
$n = 3,\ k = 1$

Mx_0 uniformly distributes on the unit sphere in $\mathbb{R}^3$.



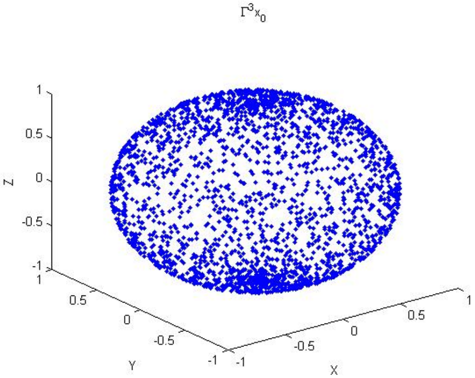
$n = 3, \ k = 2$

$M^2 x_0$ tends to sit closer to the "North Pole", x_0.



$n = 3,\ k = 3$

$M^3 x_0$ has higher density in both polar regions.



$\mathbb{P}_n[x_0' M^2 x_0 > 0]$

Dimension n	3	4	5	6
$\mathbb{P}_n[x_0' M^2 x_0 > 0]$	0.707	0.682	0.664	0.651
Dimension n	8	10	20	100
$\mathbb{P}_n[x_0' M^2 x_0 > 0]$	0.632	0.619	0.586	0.540

Table: The probabilities $\mathbb{P}_n[x_0' M^2 x_0 > 0]$ in different dimension n.

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- M_{11} and $-M_{11}$ should have the same distribution due to the symmetry, the probability density function of V_1 is,

$$f_n(x) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}(1 - x^2)^{(n-3)/2}, \qquad (1)$$

while x ranges in $-1 \leq x \leq 1$ and Γ is the gamma function.

Distribution of $x_0' M^2 x_0$

- We will partition the matrix M into the following parts,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \qquad (2)$$

where $M_{11} \in [-1, 1]$, $M_{12} \in [-1, 1]^{1\times(n-1)}$, $M_{21} \in [-1, 1]^{(n-1)\times 1}$ and $M_{22} \in [-1, 1]^{(n-1)\times(n-1)}$.

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- Then V_2 can be written as,

$$V_2 = x_0' M^2 x_0 = M_{11}^2 + M_{12} M_{21}, \tag{3}$$

Distribution of $x_0' M^2 x_0$

- We can further present V_2 as,

$$V_2 = M_{11}^2 + (1 - M_{11}^2) \frac{M_{12}}{(1 - M_{11}^2)^{1/2}} \frac{M_{21}}{(1 - M_{11}^2)^{1/2}}. \tag{4}$$

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- The motivation for doing this is to normalize both M_{12} and M_{21} into norm one, so that the part $\frac{M_{12}}{(1-M_{11}^2)^{1/2}} \frac{M_{21}}{(1-M_{11}^2)^{1/2}}$ in the equation above can be seen as an inner product of two unit vectors in $\mathbb{R}^{(n-1)}$.

Distribution of $x_0' M^2 x_0$

Let Π and Δ be two fixed $n \times n$ orthogonal matrices, which has the form

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & \Pi_1 \end{pmatrix}, \Delta = \begin{pmatrix} 1 & 0 \\ 0 & \Delta_1 \end{pmatrix}, \tag{5}$$

while Π_1 and Δ_1 are $(n-1) \times (n-1)$ orthogonal matrices. Then,

$$\Pi M \Delta = \begin{pmatrix} 1 & 0 \\ 0 & \Pi_1 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Delta_1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12}\Delta_1 \\ \Pi_1 M_{21} & \Pi_1 M_{22} \Delta_1 \end{pmatrix}. \tag{6}$$

Distribution of $x_0' M x_0$

Due to the properties of orthogonal matrices, M and $\Pi M \Delta$ should share the same distribution. This is because that the elements in $\mathcal{O}_n$ is one-to-one corresponding to the elements in $\Pi \mathcal{O}_n \Delta$, while $\mathcal{O}_n$ is the group of $n \times n$ orthogonal matrices. Then V_2 can also be expressed in the following way,

$$V_2 = x_0' \Pi M \Delta \Pi M \Delta x_0 \tag{7}$$

$$= M_{11}^2 + (1 - M_{11}^2) \frac{M_{12}}{(1 - M_{11}^2)^{1/2}} \Delta_1 \Pi_1 \frac{M_{21}}{(1 - M_{11}^2)^{1/2}}. \tag{8}$$

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$$V_2 = M_{11}^2 + (1 - M_{11}^2) \frac{M_{12}}{(1 - M_{11}^2)^{1/2}} \Delta_1 \Pi_1 \frac{M_{21}}{(1 - M_{11}^2)^{1/2}}.$$

Notice that this equation holds for all fixed Δ_1 and Π_1. Thus, for any random orthogonal matrices Δ_1 and Π_1, it should still hold, since M and $\Pi M \Delta$ have the same distribution.Then we can choose Δ_1 and Π_1 to be independent uniform on $\mathcal{O}_{(n-1)}$, so that $\Delta_1 \Pi_1$ is again uniform on $\mathcal{O}_{(n-1)}$.

Distribution of $x_0' M x_0$

Let $u = \frac{M_{12}}{(1-M_{11}^2)^{1/2}}$ and $v = \frac{M_{21}}{(1-M_{11}^2)^{1/2}}$. As we already stated above, u' and v are both unit vectors in $\mathbb{R}^{(n-1)}$. Then apply the following lemma, we can conclude that, the probability density function for

$$\frac{M_{12}}{(1-M_{11}^2)^{1/2}} \Delta_1 \Pi_1 \frac{M_{21}}{(1-M_{11}^2)^{1/2}}$$

must be $f_{n-1}(\cdot)$, as we defined in calculating V_1.
Lemma. If u and v are fixed unit vectors in $\mathbb{R}^n$, and Q is uniform on $\mathcal{O}_n$, then the density function for $u'Qv$ is $f_n(\cdot)$.

Distribution of $x_0' M x_0$

In the n dimensional space, for M be uniform distributed random orthogonal matrices, $V_2 = x_0' M^2 x_0$ behaviors as,

$$V_2 = T + (1 - T)Y,$$

where T and Y are two independent random variables. Furthermore, T obeys to the Beta distribution with parameters $\alpha = \frac{1}{2}$ and $\beta = \frac{n-1}{2}$, and the random variable Y has density function f_n^Y satisfying

$$f_n^Y(x) = f_{n-1}(x) = \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2} - 1)}(1 - x^2)^{(n-4)/2}.$$