Eigenvalues of Random Graphs

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May 2012

Random graphs

Question

What is the limiting spectral distribution of a random graph?

Random graphs

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What is the limiting spectral distribution of a random graph?

- Erdős-Rényi random graphs $G(n, p)$: edges added independently with probability p.
- Random d-regular graph $G_{n,d}$.

Key difference: edges of $G_{n,d}$ are not independent.

Eigenvalues of random graphs

Random d-regular graph $G_{n,d}$

- Largest eigenvalue is d
- All other eigenvalues are $O(\sqrt{d})$.

$G(n, p)$

- Largest eigenvalue $\approx np$
- All other eigenvalues are $O(\sqrt{np})$.

Eigenvalues of random graphs

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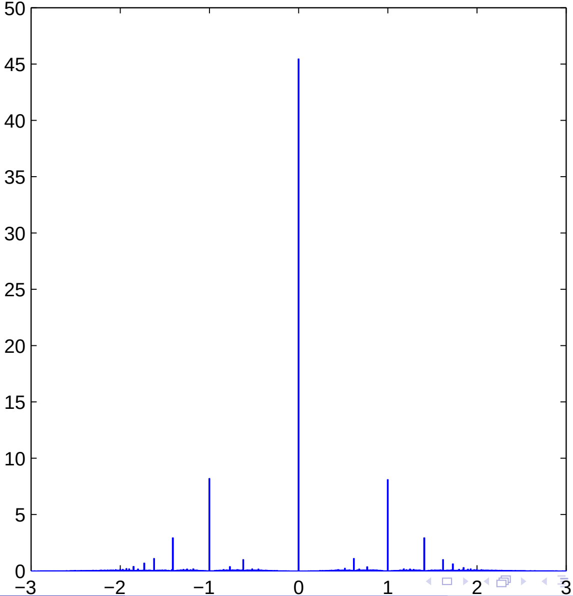
- All other eigenvalues are $O(\sqrt{np})$.

Note: In spectra plots, the matrices de-meaned and normalized.

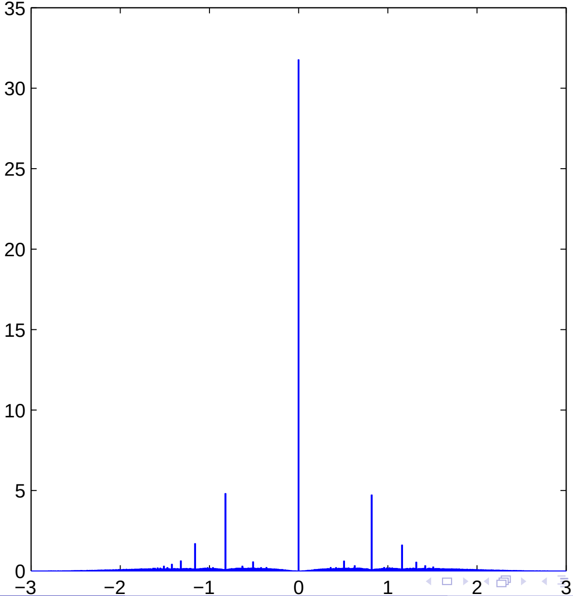
Fact: If J is rank 1, then the eigenvalues of A and $A - J$ interlace.

So shape of limiting distribution is unchanged.

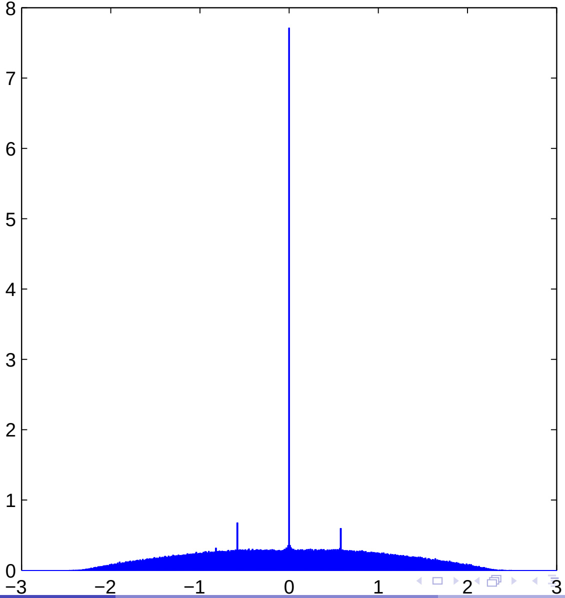
$G(n, \frac{\alpha}{n})$ when $\alpha = 1.0$



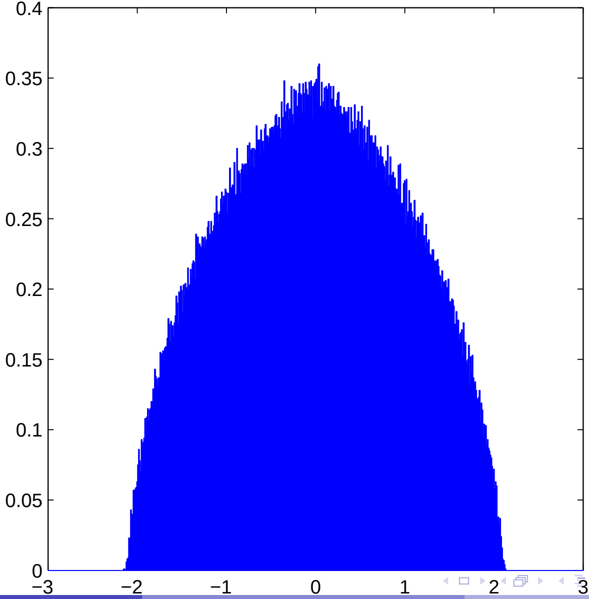
$G(n, \frac{\alpha}{n})$ when $\alpha = 1.5$



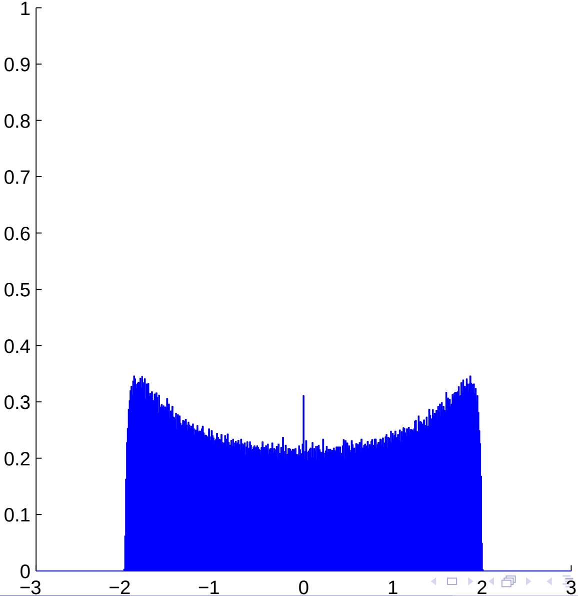
$G(n, \frac{\alpha}{n})$ when $\alpha = 3$



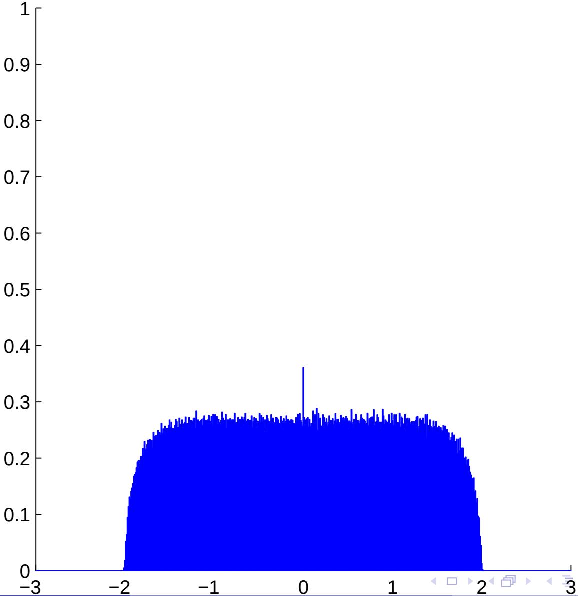
$G(n, \frac{\alpha}{n})$ when $\alpha = 10$



Random 3-regular graph



Random 6-regular graph



$G(n,p)$ when $p = \omega(1/n)$

Theorem

Let $p = \omega(\frac{1}{n})$, $p \le \frac{1}{2}$. The normalized spectral distribution of $G(n,p)$ approaches the semicircle law.

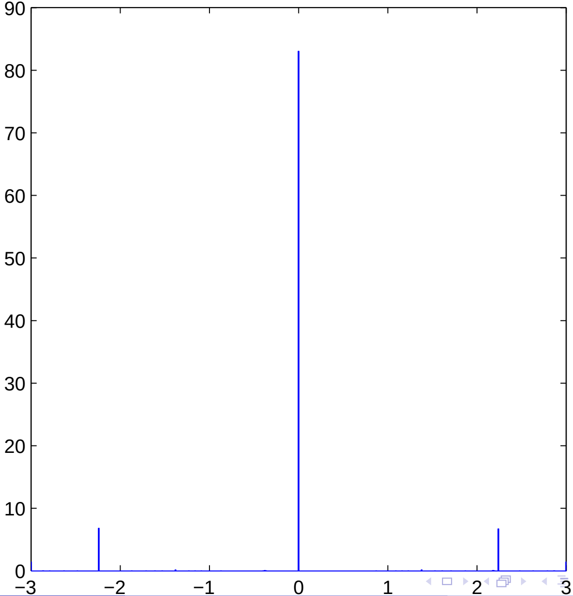
$G(n, p)$ when $p = \omega(1/n)$

Theorem

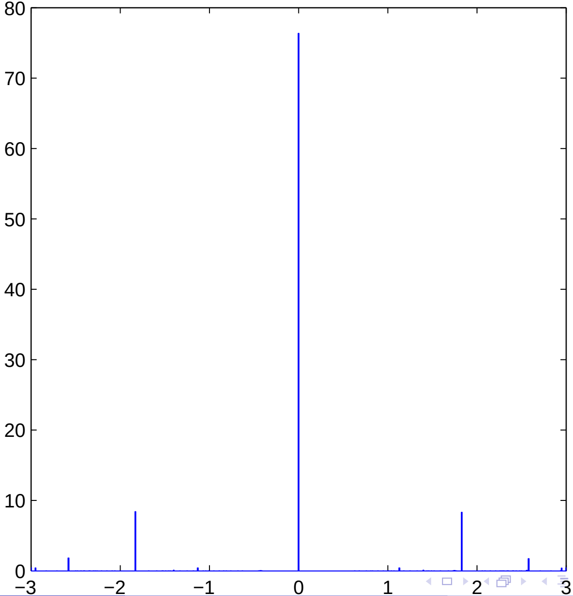
Let $p = \omega(\frac{1}{n})$, $p \leq \frac{1}{2}$. The normalized spectral distribution of $G(n, p)$ approaches the semicircle law.

- Proof is basically same as Wigner's theorem.
- Method of moments.
- Counting walks and trees. Catalan numbers.

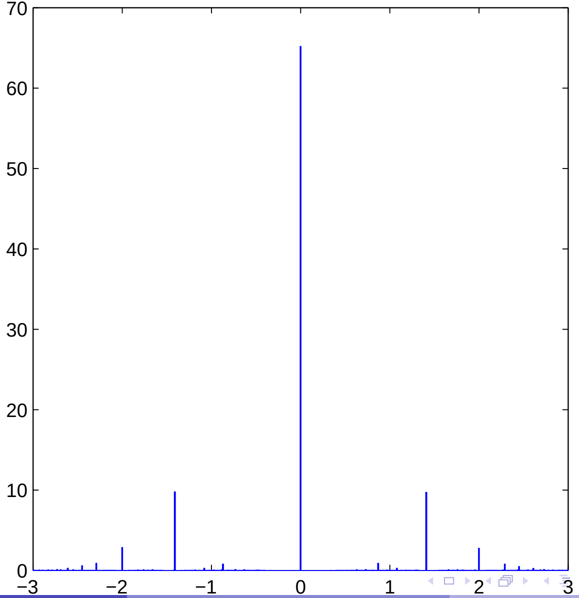
$G(n, \frac{\alpha}{n})$ when $\alpha = 0.2$



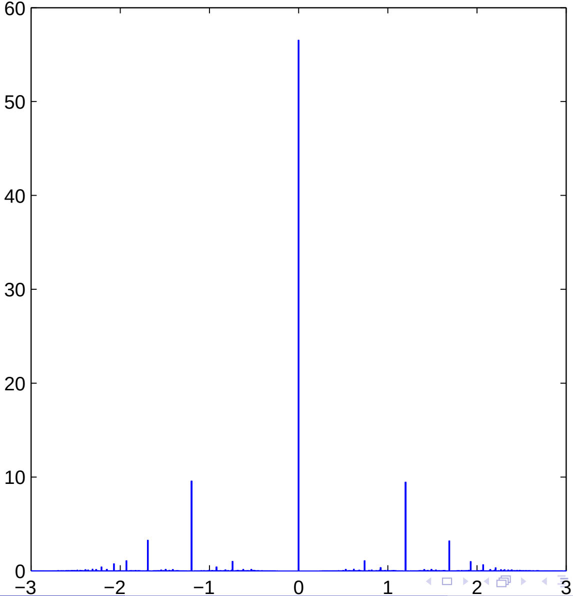
$G(n, \frac{\alpha}{n})$ when $\alpha = 0.3$



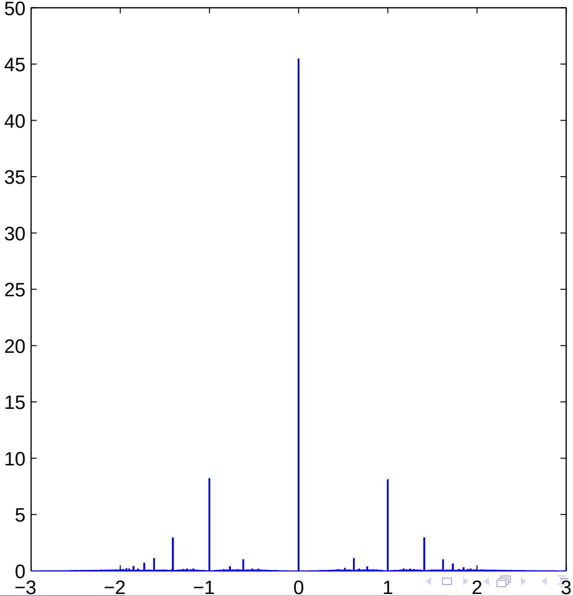
$G(n, \frac{\alpha}{n})$ when $\alpha = 0.5$



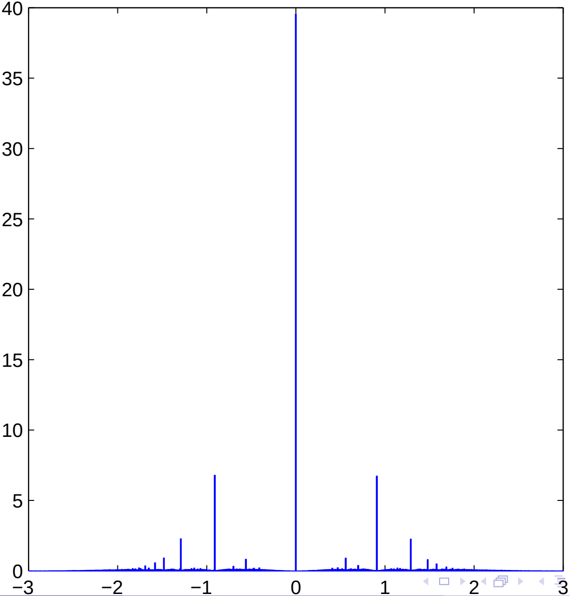
$G(n, \frac{\alpha}{n})$ when $\alpha = 0.7$



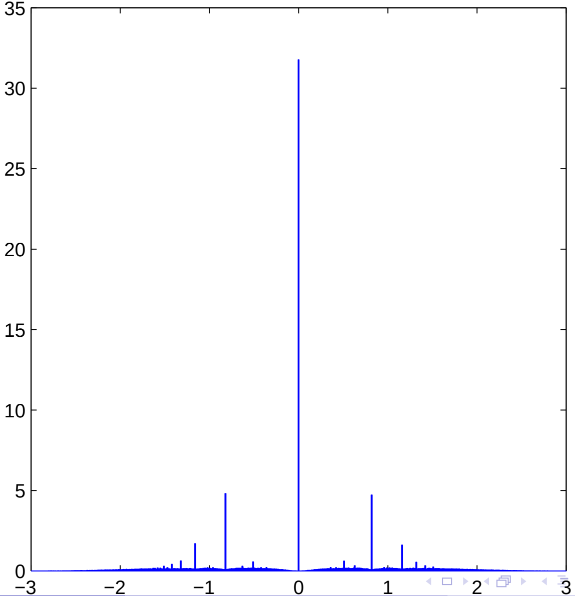
$G(n, \frac{\alpha}{n})$ when $\alpha = 1.0$



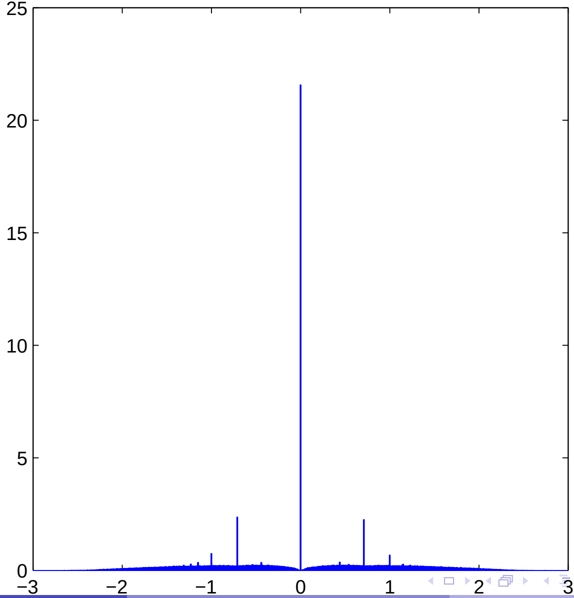
$G(n, \frac{\alpha}{n})$ when $\alpha = 1.2$



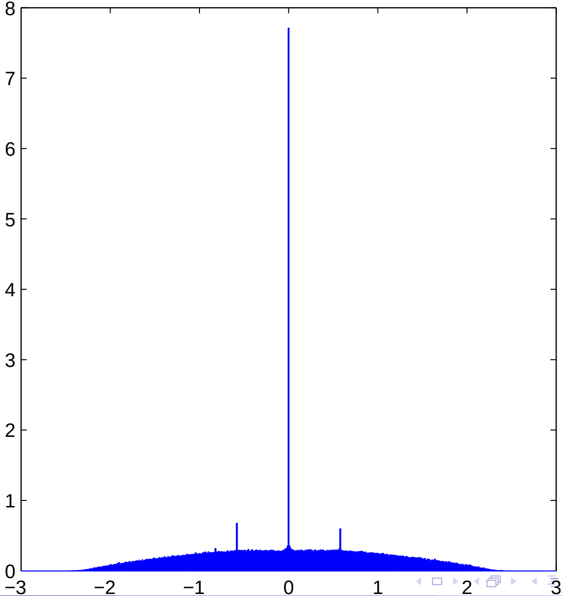
$G(n, \frac{\alpha}{n})$ when $\alpha = 1.5$



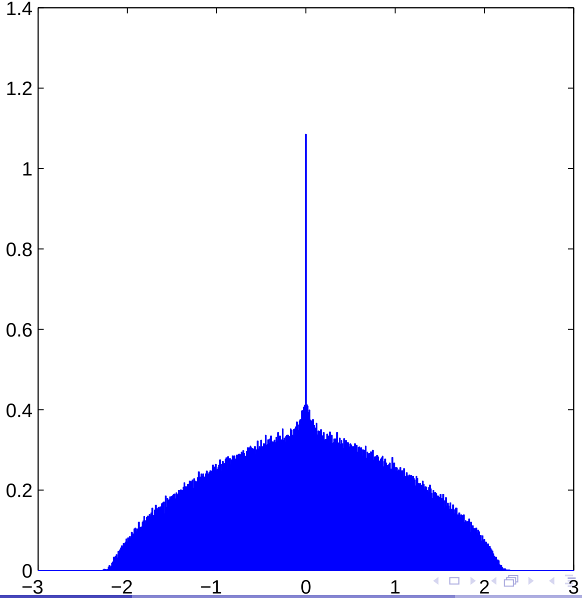
$G(n, \frac{\alpha}{n})$ when $\alpha = 2$



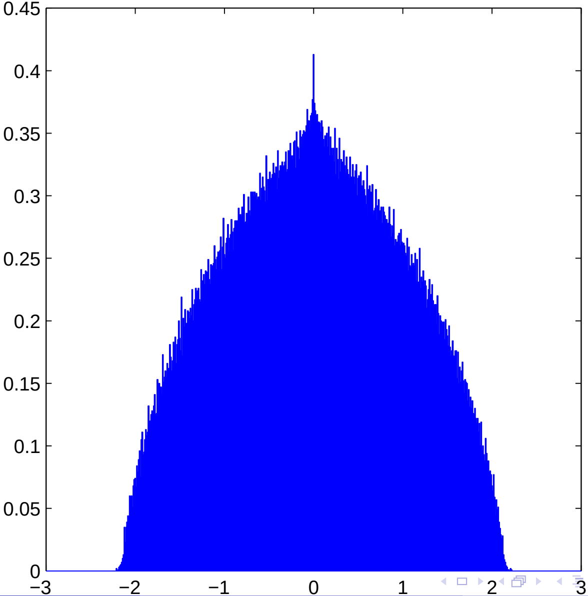
$G(n, \frac{\alpha}{n})$ when $\alpha = 3$



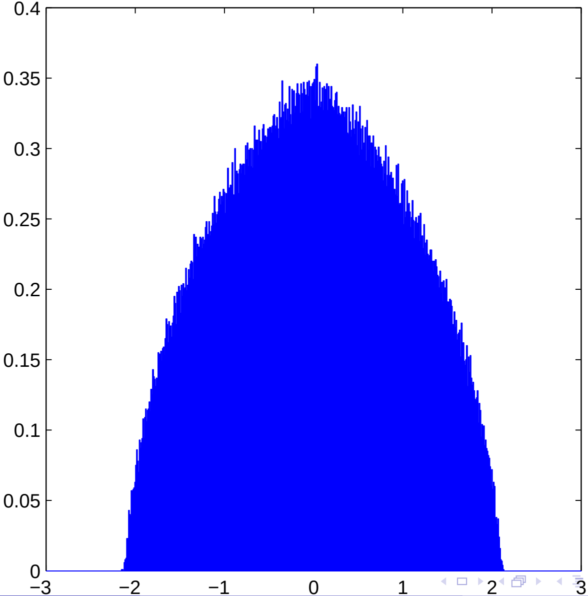
$G(n, \frac{\alpha}{n})$ when $\alpha = 5$



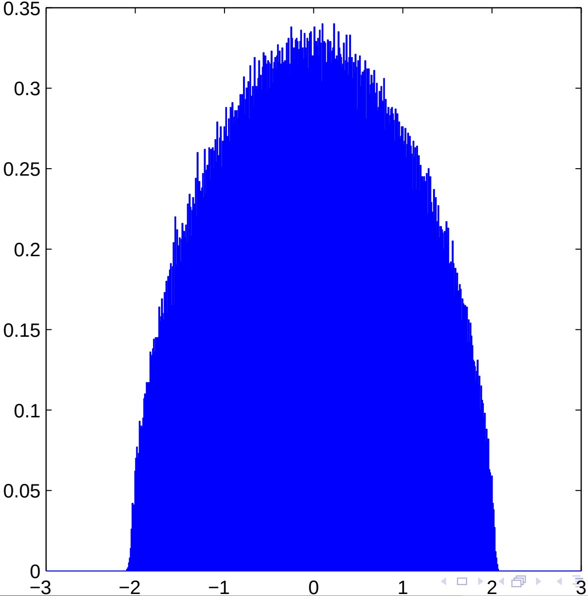
$G(n, \frac{\alpha}{n})$ when $\alpha = 7$



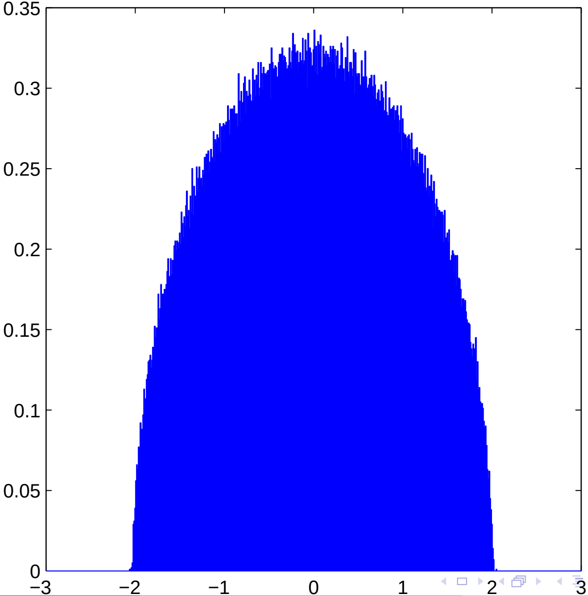
$G(n, \frac{\alpha}{n})$ when $\alpha = 10$



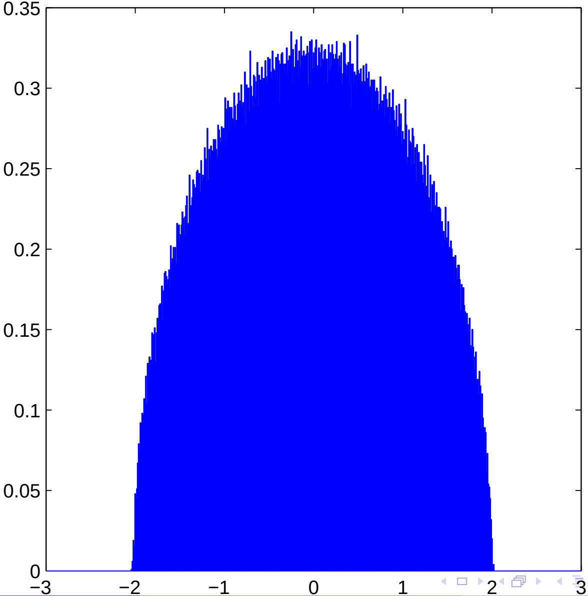
$G(n, \frac{\alpha}{n})$ when $\alpha = 20$



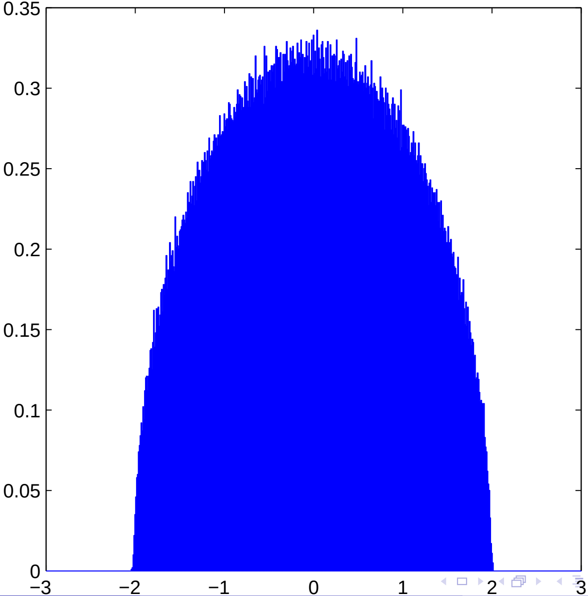
$G(n, \frac{\alpha}{n})$ when $\alpha = 50$



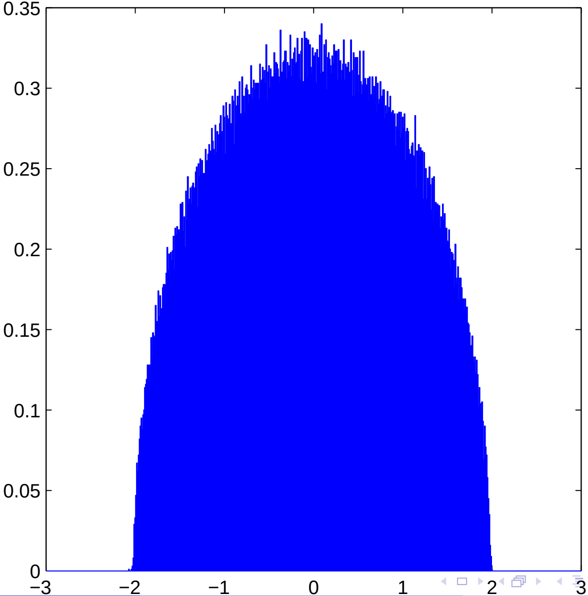
$G(n, \frac{\alpha}{n})$ when $\alpha = 70$



$G(n, \frac{\alpha}{n})$ when $\alpha = 100$



$G(n, \frac{\alpha}{n})$ when $\alpha = 200$



$G(n, \frac{\alpha}{n})$

- Discrete component — spikes
- Continuous component
- To explain this phenomenon, we need to understand the structure of $G(n, p)$.

Structure of a random graph

P. Erdős and A. Rényi. *On the evolution of random graphs.* 1960.

Structure of $G(n,p)$, almost surely for n large:

- $p = \frac{\alpha}{n}$ with $\alpha < 1$.

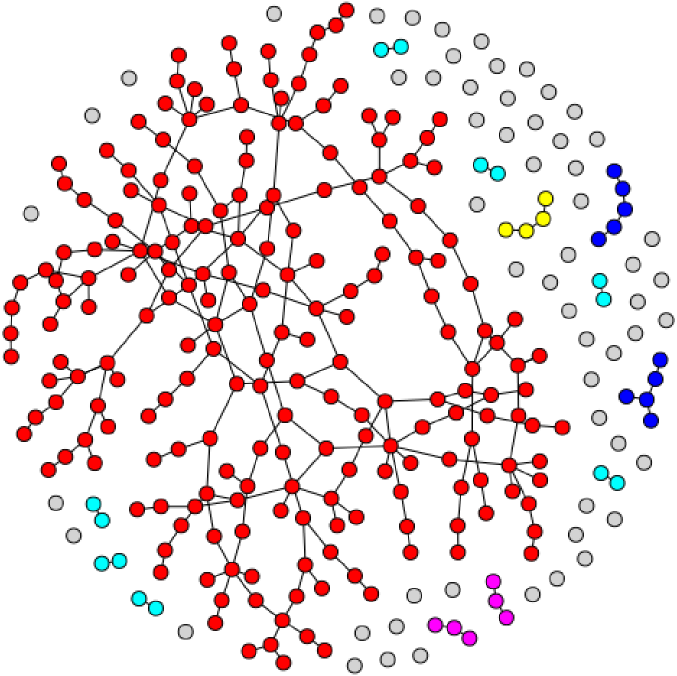
  All components have small size $O(\log n)$, mostly trees.

- $p = \frac{\alpha}{n}$ with $\alpha = 1$.

  Largest component has size on the order of $n^{2/3}$.

- $p = \frac{\alpha}{n}$ with $\alpha > 1$,

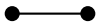
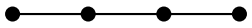
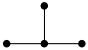
  One giant component of linear size; and all other components have small size $O(\log n)$, mostly trees.



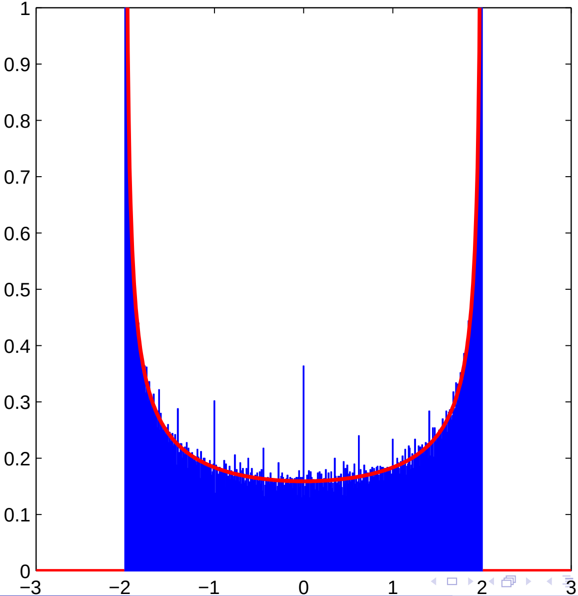
Spectra of $G(n, \frac{\alpha}{n})$

- Properties of spectra: very few rigorous proofs; lots of intuition and "physicists' proofs".
- Continuous spectrum + discrete spectrum
- Suspected that the giant component contributes to the continuous spectrum
- and isolated and hanging trees contribute to the discrete spectrum.

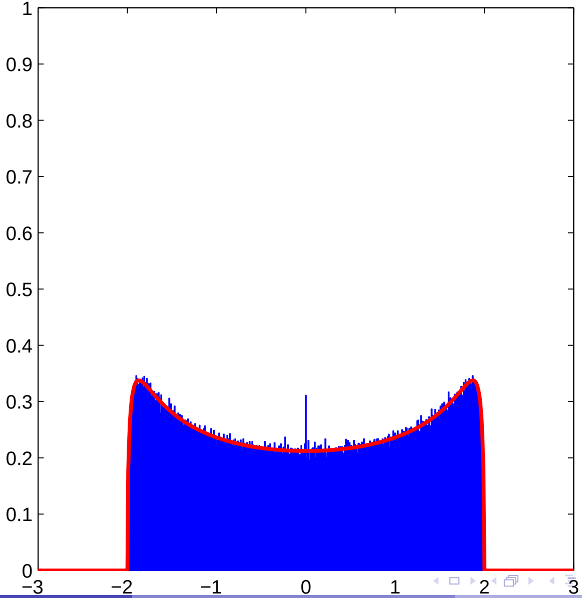
Trees give the spikes

T	$A(T)$	Eigenvalues
	$\begin{pmatrix} 0 \end{pmatrix}$	0
	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$-1,\ 1$
	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$-\sqrt{2}$, 0, $\sqrt{2}$ (-1.41) (1.41)
	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\frac{-1-\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$, $\frac{-1+\sqrt{5}}{2}$, $\frac{1+\sqrt{5}}{2}$ (-1.62) (-0.62) (0.62) (1.62)
	$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$-\sqrt{3}$, 0, 0, $\sqrt{3}$ (-1.73) (1.73)

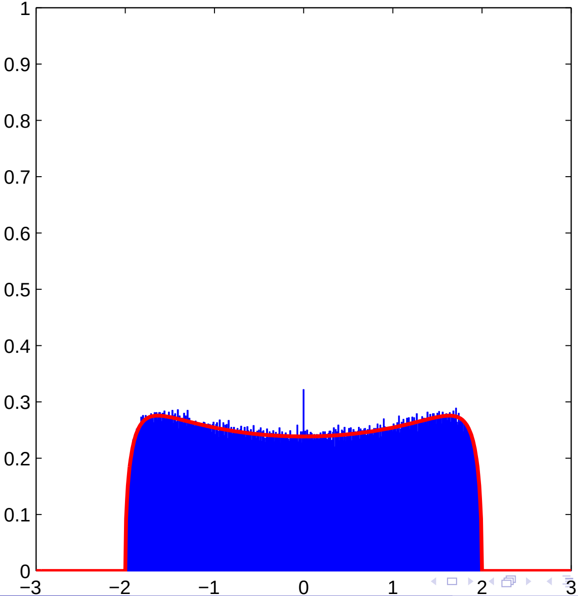
Random 2-regular graph



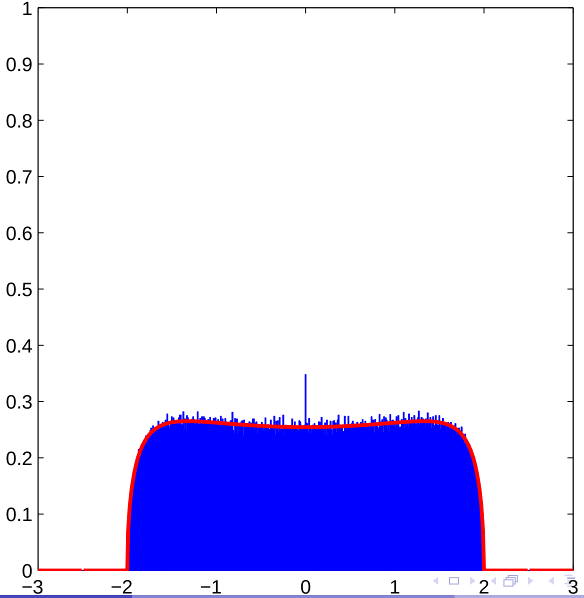
Random 3-regular graph



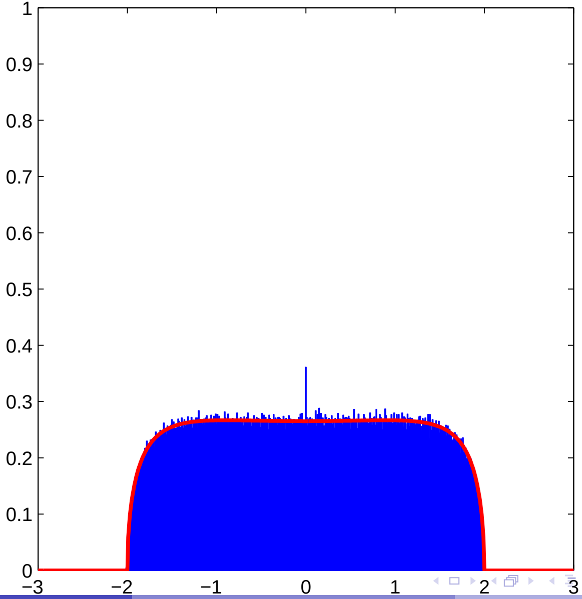
Random 4-regular graph



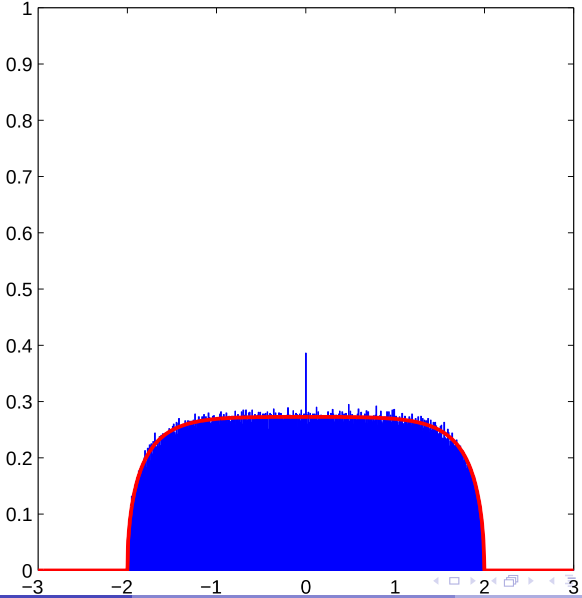
Random 5-regular graph



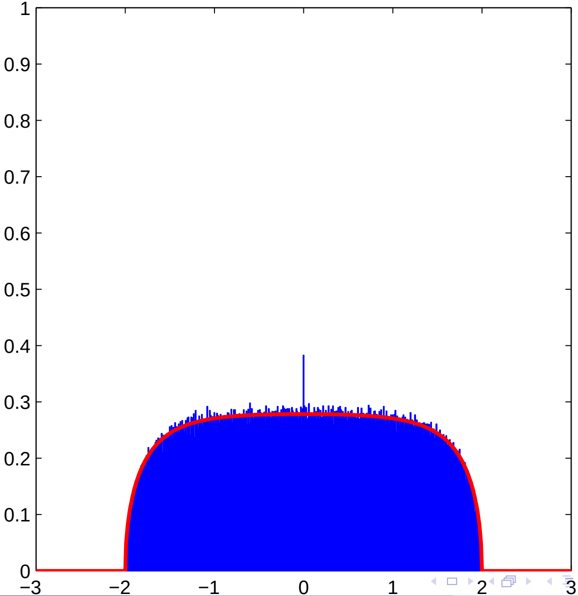
Random 6-regular graph



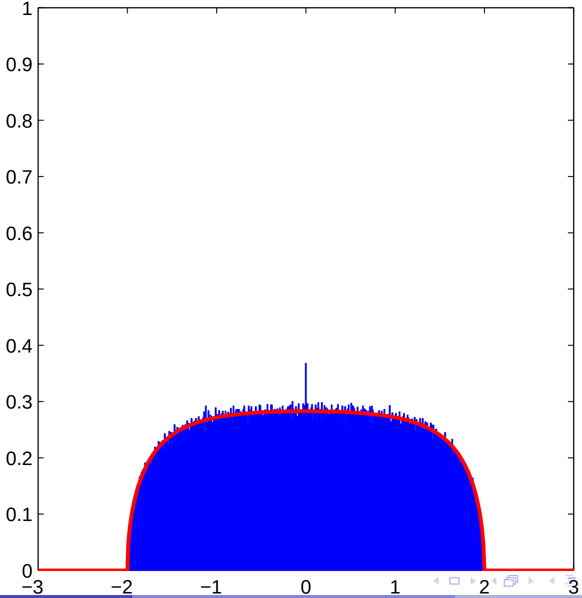
Random 7-regular graph



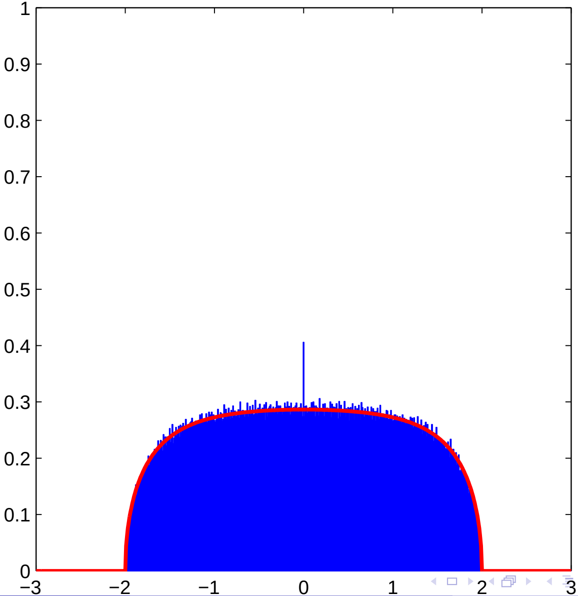
Random 8-regular graph



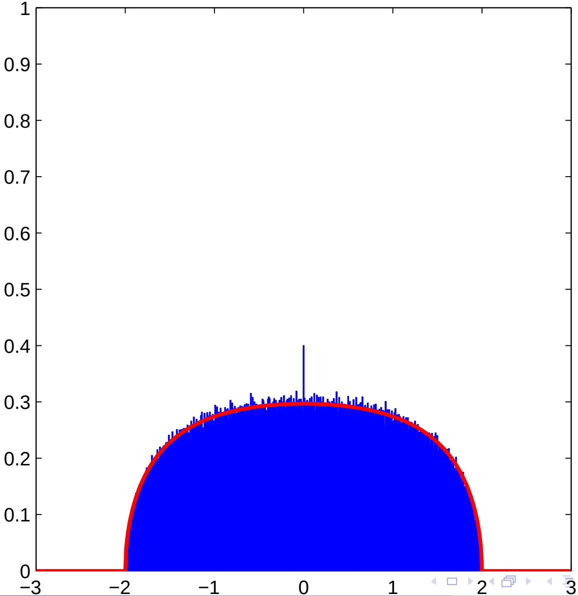
Random 9-regular graph



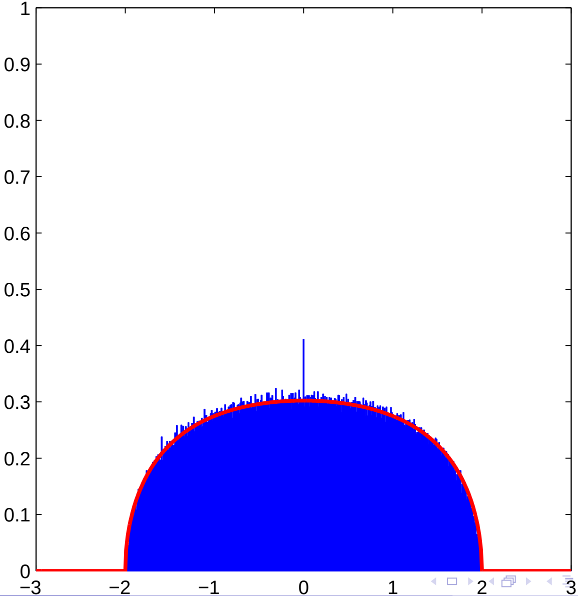
Random 10-regular graph



Random 15-regular graph



Random 20-regular graph



Random d-regular graphs

Theorem (McKay 1981)

Let $d \geq 2$ be a fixed integer. As $n \to \infty$, the spectral distribution of a random d-regular graph $G_{n,d}$ on n vertices approaches

$$f_d(x) = \begin{cases} \dfrac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}, & \text{if } |x| \leq 2\sqrt{d-1}; \\ 0, & \text{otherwise.} \end{cases}$$

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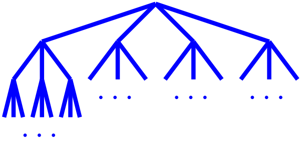
Proof idea:

- Method of moments.
- Reduce to counting closed walks in $G_{n,d}$.
- Locally $G_{n,d}$ looks like a d-regular tree.



Random d-regular graphs with d growing

Theorem (Tran-Vu-Wang 2012)

Let $d \to \infty$, $d \le \frac{n}{2}$. As $n \to \infty$, the spectral distribution of a random d-regular graph $G_{n,d}$ on n vertices converges to the semicircle distribution.

Random d-regular graphs with d growing

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Proof idea:

- $G(n, \frac{d}{n})$ is d-regular with some (small) probability
- But the probability that the spectral distribution of $G(n, \frac{d}{n})$ deviates from the semicircle is even smaller.
- So with high probability the spectral distribution of $G_{n,d}$ is close to the semicircle.

Summary

Erdős-Rényi random graph $G(n,p)$

- $p = \frac{\alpha}{n}$: observed continuous + discrete spectrum
- $p = \omega(\frac{1}{n})$: semicircle [Wigner 1955]

Random d-regular graph

- Fixed d: [McKay 1981]
- Growing d: semicircle [Tran-Vu-Wang 2012]