18.338 Project
Spectral Distributions of Random Graphs

Yufei Zhao

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1 Introduction

Given a graph G with n vertices, its adjacency matrix $A(G)$ is the $n \times n$ matrix whose (i, j) entry is 1 if vertices i and j are adjacent, and 0 otherwise. The eigenvalues of the graph G are defined to be the eigenvalues of $A(G)$. The collection of eigenvalues of G are also known as the *spectrum* of G. In this project, we investigate the empirical eigenvalue distribution (ESD) of a random graph.

In class, we considered ensembles of random matrices where the entries of the matrix are independent and identically distributed. We are often interested in the limit of the ESD as $n \to \infty$. The most famous result of this type is Wigner's semicircle law [8, 9]. The random matrices in Wigner's setup correspond to random graphs where each possible edge appears independently with some probability p. This random graph is denoted $G(n, p)$, and it was first studied in the seminal paper of Erdős and Rényi [2]. If p were held constant as n grows, then the setup is equivalent to Wigner's random matrix setting. However, in graph theory, we are often interested in cases where p decreases with n. In Section 2, we discuss some results and observations relating to the limiting ESD of $G(n, p)$.

In Section 3, we consider another model of random graphs. A *d-regular graph* is a graph where every vertex has degree d. Let $G_{n,d}$ denote a random d-regular graph on n vertices, chosen uniformly at random from all d-regular graphs on n labeled vertices[1]. Unlike the Erdős-Rényi random graph, the edges of $G_{n,d}$ are not independent. It turns out that if d is fixed, then the limiting ESD of $G_{n,d}$ is not the semicircular distribution. McKay [4] determined the limiting ESD for each d. Very recently, Tran, Vu, and Wang showed that if d increases with n, then the limiting ESD of $G_{n,d}$ does approach the semicircle distribution. We discuss both results.

In this project, we focus on the spectral distribution of graphs. There are many other questions about eigenvalues of random graphs that could be asked, e.g., what is the distribution of the largest eigenvalue, and what is the distribution of the spectral gap (the difference between the top two eigenvalues). Questions about largest eigenvalues are very important in graph theory, as many important parameters of graphs can be characterized by their largest eigenvalues and spectral gaps. The field of *spectral graph theory* is dedicated to the properties of graph eigenvalues and their applications. However, we shall not explore this territory in this project.

We shall discuss the above two models of random graphs separately. In Section 2 we discuss the Erdős-Rényi model $G(n, p)$ and in Section 3 we discuss the random d-regular model $G_{n,d}$. We only include sketches of proofs and refer the readers to the original papers for details. We include MATLAB plots of the spectral distributions for both models of random graphs, as Prof. Edelman has passed to us the wisdom that these plots can sometimes be even more convincing than the proofs.

2 Erdős-Rényi random graph

The Erdős-Rényi random graph $G(n, p)$ is formed by taking an empty graph on n vertices and adding each edge independently with probability p. The study of this random graph model was initiated in the seminal paper of Erdős and Rényi [2]. In this section, we discuss the ESD of $G(n, p)$. See Figure 1

[1] Note that $G_{n,d}$ makes sense only when nd is even, and we shall always assume that this is the case.

for some plots. The plots show the observed spectral distribution of the graphs after we de-mean[2] and normalize the matrices, following (1). We begin with a discussion of the case[3] $p = \omega(1/n)$, where Wigner's semicircle law still holds. However, $p = O(1/n)$, new phenomena begin to emerge, and we observe a discrete component to the spectrum of the graph. In order to discuss these observations, we will need to review the results of Erdős and Rényi about the qualitative nature of the components of $G(n, p)$.

We focus on the case when $p \le \frac{1}{2}$ since the other case can be analyzed by taking the graph complement.

2.1 $G(n, p)$ when $p = \omega(1/n)$

In the case when $p = \omega(\frac{1}{n})$ and $p \le \frac{1}{2}$, the limiting empirical spectral distribution is the semicircular distribution. It can be proved by the method of moments, and the proof is essentially the same as that of Wigner's semicircle law for random matrices. We state the result and sketch an outline of the proof. Details can be found in [1, Theorem 3.4] or [7, Appendix A].

Theorem 2.1. *Assume $p = \omega(\frac{1}{n})$ and $p \le \frac{1}{2}$. Let A_n be the adjacency matrix of a random graph $G(n, p)$. Then, as $n \to \infty$, the empirical spectral distribution of the matrix $\frac{1}{\sqrt{np(1-p)}} A_n$ converges in distribution to the semicircle distribution which has a density $\rho_{\mathrm{sc}}(x)$ with support on $[-2, 2]$,*

$$\rho_{\mathrm{sc}}(x) := \frac{1}{2\pi}\sqrt{4 - x^2}.$$

Let

$$\sigma = \sqrt{p(1-p)}$$

be the standard deviation of the non-diagonal entries of A. Observe that $\frac{1}{\sigma}A_n$ is a random matrix whose non-diagonal entries, $i < j$, are binomial random variables taking value $\frac{1}{p(1-p)}$ with probability p and 0 otherwise. Note that the variance of ζ_{ij} is 1. It will be convenient to shift the entries so that the mean is zero. Let J_n be the $n \times n$ matrix all of whose entries are 1. It is known (see [6, Lemma 39]) that the eigenvalues of $\frac{1}{\sqrt{n}\sigma}A_n$ and those of $\frac{1}{\sqrt{n}\sigma}(A_n - pJ_n)$ interlace, so they share the same global spectral properties, and in particular their limiting ESDs are identical. So instead of working with $\frac{1}{\sigma}A_n$, we consider the de-meaned matrix

$$M_n = \frac{1}{\sigma}(A_n - pJ_n) \tag{1}$$

whose non-diagonal entries ξ_{ij}, $i < j$, are binomial random variables with mean 0 and variance 1, taking value $\frac{1-p}{\sigma}$ with probability p and value $\frac{-p}{\sigma}$ with probability $1 - p$. Note that when $p = \omega(\frac{1}{n})$, $\left|\xi_{ij}\right| = o(\sqrt{n})$. We can now finish the proof essentially the same way as the method of moments proof of Wigner's semicircle law.

[2]The spectrum of $G(n, p)$ almost surely has a single eigenvalue around np, with all other eigenvalues $O(\sqrt{np})$. By de-meaning the matrix, we get rid of lone large eigenvalue, which makes the spectra easier to plot. The shape of the limiting distribution remains unaffected, as explain in the paragraph before (1).

[3]We use the following standard asymptotic notations: as $n \to \infty$, $f(n) = O(g(n))$ means that there exists some C so that $\left|f(n)\right| \le Cg(n)$ for n sufficiently large; $f(n) = o(g(n))$ means that $f(n)/g(n) \to 0$; $f(n) = \Omega(g(n))$ means that there exists some $c > 0$ so that $f(n) \ge cg(n)$ for n sufficiently large; $f(n) = \omega(g(n))$ means that $f(n)/g(n) \to \infty$; and finally $f(n) = \Theta(g(n))$ means that $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ simultaneously.

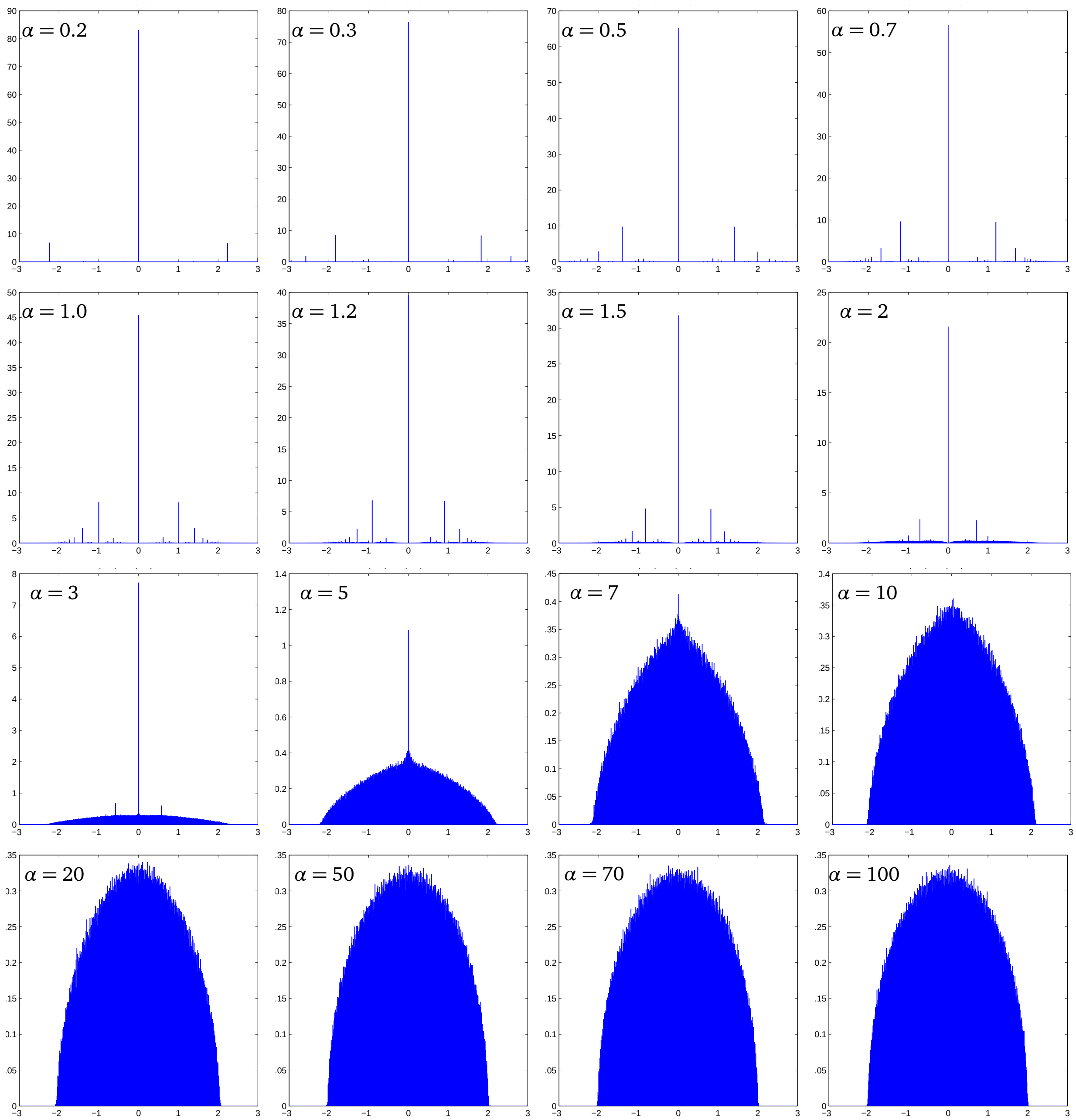


Figure 1: Normalized empirical spectral distribution of $G(n, \frac{\alpha}{n})$ for various values of α. Taken with $n = 1000$ using 100 trials.

2.2 $G(n,p)$ when $p = \alpha/n$

When $p = O(\frac{1}{n})$, the empirical spectral distribution of $G(n,p)$ no longer seems to converge to semicircle distribution. Let us consider the case when $p = \frac{\alpha}{n}$ and $n \to \infty$. See Figure 1 the observed (normalized) eigenvalue distribution for $G(n, \frac{\alpha}{n})$ when $n = 1000$ and for various values of α. We make the following observations:

1. The spectra seems to be a composition of two components: a "discrete component" consisting of spikes, and a "continuous component."
2. For small values of α, the discrete spectrum is dominant, and for larger values of α, the continuous spectrum is dominant.
3. The continuous spectrum seems to be approaching a semicircle for as α gets larger.

The third point makes sense from the result in the previous section, since we know that if $\alpha \to \infty$, however slowly with n, then the limiting distribution is indeed a semicircle distribution. To explain the presence of the discrete spectra, we need some information about the structure of $G(n,p)$, which we discuss next.

2.3 Evolution of a random graph

One of the key results of Erdős and Rényi concerns the qualitative nature of the structure of $G(n,p)$ for different regimes of p. The graph breaks into a number of connected components. When $p = \frac{\alpha}{n}$, with α fixed and $n \to \infty$, the size of the largest connected component has the following "double jump" behavior:

- When $p = \frac{\alpha}{n}$ with $\alpha < 1$, almost surely all components of $G(n,p)$ will have size $O(\log n)$, mostly being trees.
- When $p = \frac{1}{n}$, almost surely the largest component of $G(n,p)$ has size on the order of $n^{2/3}$.
- When $p = \frac{\alpha}{n}$ with $\alpha > 1$, almost surely there a unique largest connected component (the giant component) of size $g(\alpha)n$, where g is some continuous function satisfying $g(\frac{1}{2}) = 0$ and $\lim_{\alpha\to\infty} g(\alpha) = 1$. All other componented have size $O(\log n)$, mostly trees.

In other words, $p = \frac{1}{n}$ is a sharp threshold for the existence of a giant component in $G(n,p)$. It is also known that $p = \frac{\log n}{n}$ is a sharp threshold for $G(n,p)$ being connected.

The above characterization of the structure of $G(n,p)$ helps us to make an attempt at explaining the spectra observed in the previous section. Note that most of these claims are speculative in nature, as no rigorous proofs are known (although some "physicist's proofs" exist). See the Princeton senior thesis of Spiridonov [5] for more discussion. We recall the standard fact that the spectra of a graph can be formed by combining (i.e., taking a multiset union) the spectra of the connected components of a graph.

- When $p = \frac{\alpha}{n}$ with $\alpha < 1$, the spectrum is contributed entirely by trees. We should be able to approximate the limiting ESD by computing the spectra of small trees. The dominance of trees explains the discrete nature of the spectra.
- When $p = \frac{\alpha}{n}$ with $\alpha > 1$, there is a giant component, which contributes to the continuous component of the spectra. There are still small connected components, mostly trees, that contribute to the discrete component of the spectra. Also contributing to the observed discrete spectra are the trees with one leaf vertex attached to the giant component.

In other words, it is suspected that the discrete component of the spectra, i.e., the spikes in the eigenvalue histograms, come from small trees. Shown in Table 1 are a number of small trees along with their eigenvalues. The values seem to coincide with the dominant spikes in the observed spectra. The height of the spikes depend on the α, with the smaller trees having greater contribution.

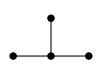
T	$A(T)$	Eigenvalues
	$\begin{pmatrix} 0 \end{pmatrix}$	0
	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$-1,\ 1$
	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$-\sqrt{2}$, 0, $\sqrt{2}$ (-1.41) (1.41)
	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\frac{-1-\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$, $\frac{-1+\sqrt{5}}{2}$, $\frac{1+\sqrt{5}}{2}$ (-1.62) (-0.62) (0.62) (1.62)
	$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$-\sqrt{3}$, 0, 0, $\sqrt{3}$ (-1.73) (1.73)

Table 1: Eigenvalues of some small trees

3 Random regular graphs

In this section, we consider a different random graph model from the previous section. Recall that a d-regular graph is a graph where every vertex has degree d. Let $G_{n,d}$ be a random d-regular graph, where we choose uniformly at random from all d-regular graphs on n labeled vertices. Note that $G_{n,d}$ and $G(n, \frac{d}{n-1})$ have the same edge density, but a key difference is the entries of the adjacency matrix of $G(n, \frac{d}{n-1})$ are independent, while those of $G_{n,d}$ are not.

Here is simple algorithm for generating $G_{n,d}$, which is also what we use for the numerical experiments. Start with an empty graph on n vertices, and draw d "stubs" from each vertex, where each stub is an edge with one endpoint already attached to a vertex and the other endpoint free. At each step, glue together two uniformly random free ends of stub-edges (disallowing the gluing if this would create a loop or a multi-edge). At the end of the process[4], when there are no free ends remaining, we obtain a random d-regular graph $G_{n,d}$.

3.1 Constant degree — McKay law

When d is fixed, and $n \to \infty$, the limiting empirical spectral distribution of is not the semicircle distribution. The limiting ESDs were found by McKay [4]. These limiting distributions are sometimes known

[4]There is a neglibly small probability that the process gets stuck at the end when all remaining stubs eminate from the same vertex, in which case we can just restart.

as the Kesten–McKay distributions, as the distributions were previously discovered by Kesten [3] in the context of random walks on groups. Note these distributions did not arise in the classical settings of random matrix theory. See Figure 2 for some plots. As with Figure 2, the plotted spectra are demeaned and normalized as $M_n = \frac{1}{\sqrt{d-1}}(A_n - \frac{d}{n}J_n)$.

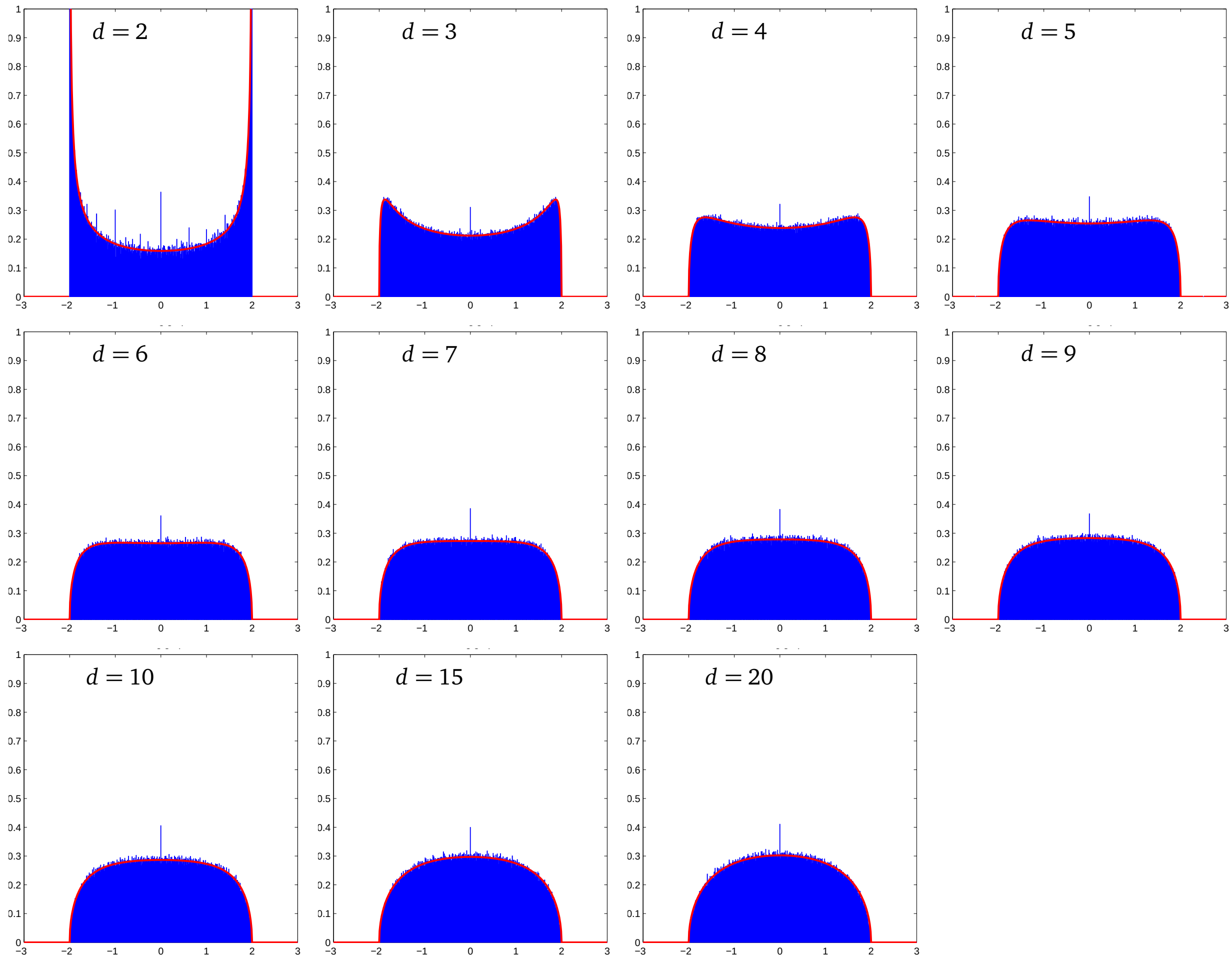


Figure 2: Normalized empirical spectral distribution of a random d-regular graph $G_{n,d}$ for various values of d. Taken with $n = 1000$ using 100 trials.

Theorem 3.1. *Let $d \geq 2$ be a fixed integer. As $n \to \infty$, the empirical spectral distribution of a random d-regular graph on n vertices approaches*

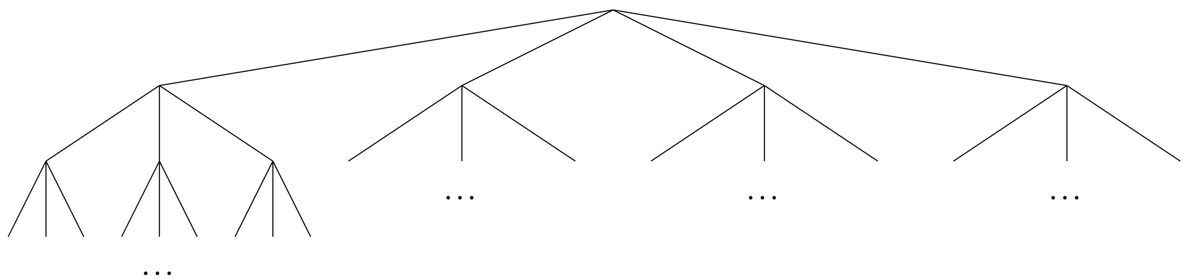
$$f_d(x) = \begin{cases} \dfrac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}, & \text{if } |x| \leq 2\sqrt{d-1}; \\ 0, & \text{otherwise.} \end{cases}$$

Here we give a sketch of the proof of Theorem 3.1, following the idea of the original paper of McKay[4] but with a somewhat different combinatorial analysis. The idea is to use the method of moments to count trees, similar to the proof of Wigner's semicircular result.

Fix d. Let $A_n = (a_{ij})$ denote the adjacency matrix of $G_{n,d}$. Let m_k denote the k-th moment of the limiting ESD, so that

$$m_k = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\,\mathrm{Tr} A_n^k.$$

Note that $\mathrm{Tr} A_n^k$ is the number of closed walks of length k in A_n. When d is fixed and $n \to \infty$, almost surely the graph is locally a d-regular tree. So in the limit, $\frac{1}{n}\mathrm{Tr} A_n^k$ is number of closed walks of length k in an (infinite) d-regular tree starting at the root. The tree for $d = 4$ is shown below.



So m_k is the number of closed walks in a d-regular tree starting at the root. Since the length of a closed walk on a tree is always even, we have $m_k = 0$ whenever k is odd. We now use techniques from enumerative combinatorics to determine a formula (specifically, a generating function) for m_{2k}.

In a walk of length $2k$, suppose the walk returns to the root for the first time after $2(i+1)$ steps, for $0 \le i \le k-1$. The first step is a down-step to the first level (we call the root the zeroth level), and there are d choices for the first step. Subsequently, before returning to the root, the walk takes place below the first level. At each step it takes either a down-step (in which there are always $d-1$ choices) or it has an up-step. Between steps 1 and $2i+1$, we have a closed walk staying below the first level, and thus the sequence of choices for up-step or down-step corresponds to a Dyck path, and the number of such choices is the i-th Catalan number C_i. Since there are $d-1$ choices for each down step, the number of possible walks between steps 1 and $2i+1$ is $C_i(d-1)^i$. After the walk returns to the root for the first time, there are now $m_{2(k-1-i)}$ ways to continue. So we obtain the recurrence relation

$$m_{2k} = d\sum_{i=0}^{k-1} C_i(d-1)^i m_{2(k-1-i)}.$$

We turn this recurrence relation into a relation of generating functions in the variable y. Multiplying by y^k and summing over all y, we find that

$$\sum_{k=0}^{\infty} m_{2k}y^k = 1 + d\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} C_i(d-1)^i m_{2(k-1-i)}y^k = 1 + dy\left(\sum_{k=0}^{\infty} C_k(d-1)^k y^k\right)\left(\sum_{k=0}^{\infty} m_{2k}y^k\right).$$

Let

$$M(y) = \sum_{k=0}^{\infty} m_{2k}y^k$$

be the generating function for m_{2k}. We know from the generating function of the Catalan numbers that

$$\sum_{k=0}^{\infty} C_k(d-1)^k y^k = \frac{1-\sqrt{1-4(d-1)y}}{2(d-1)y}.$$

So we get

$$M(y) = 1 + \frac{d}{2(d-1)}\left(1-\sqrt{1-4(d-1)y}\right)M(y)$$

and thus

$$M(y) = \left(1 - \frac{d}{2(d-1)}\left(1 - \sqrt{1-4(d-1)y}\right)\right)^{-1} \tag{2}$$

To show that $f_d(x)$ is indeed the limiting distribution, by the moment method, it suffices to check that

$$\int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} x^k f_d(x)\, dx = m_k$$

for all k (we can check that Carleman's condition applies for the uniqueness of the limiting distribution). Let M_k denote the left-hand side quantity. Then $M_k = 0$ for odd k since the distribution f_d is symmetric about zero. The generating function for M_{2k} is

$$\begin{aligned}\sum_{k=0}^{\infty} M_{2k} y^k &= \sum_{k=0}^{k} \int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} x^{2k} y^k f_d(x)\, dx \\ &= \int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} \frac{f_d(x)}{1-x^2 y}\, dx \\ &= \int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)(1-x^2y)}\, dx\end{aligned}$$

The final integral can be evaluated using the following procedure: (1) substitute $x = 2\sqrt{d-1}\cos\theta$; (2) convert it to a complex contour integral along the unit circle using $z = e^{i\theta}$; (3) evaluate the integral using residue theorem. The calculation is routine but tedious, so we omit the details. The final result of the calculation show that it is the same generating function as (2), so that $M_k = m_k$ for all k, and thus the method of moments show that ESD of $G_{n,d}$ indeed converges to f_d as claimed.

3.2 Increasing degrees

In the previous section, we considered random graphs $G_{d,n}$, letting $n \to \infty$ while holding d fixed. It was recently shown by Tran, Vu, and Wang [7] that, instead of holding d fixed, if we let $d \to \infty$, however slowly with n, then the limiting ESD is the semicircle distribution, similar to the case of $G(n, p)$ when $p = \omega(\frac{1}{n})$.

Theorem 3.2 (Tran-Vu-Wang). *Let $d \to \infty$ and $d \le \frac{n}{2}$. Let A_n be the adjacency matrix of a $G_{n,d}$, and let $\sigma = \sqrt{\frac{d}{n}(1-\frac{d}{n})}$. Then, as $n \to \infty$, the empirical spectral distribution of the matrix $\frac{1}{\sqrt{n}\sigma}A_n$ converges in distribution to the semicircle distribution ρ_{sc}.*

We sketch just the main idea of the proof. It was shown that if $d \to \infty$, then $G(n, \frac{d}{n})$ is d-regular with probability at least $e^{-O(n\sqrt{d})}$. This is a small probability, but it is bounded from below. We know from Theorem 2.1 that the normalized ESD of $G(n, p)$ approaches the semicircle distribution. What Tran, Vu, and Wang showed is a quantitative version of this convergence. They proved a high-concentration result showing that probability that the ESD of $G(n, p)$ deviates, in some sense, from the semicircle distribution is even much smaller than $e^{-O(n\sqrt{d})}$, so that with high probability, a random d-regular graph also has its ESD close to a semicircle.

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