KP Hierarchy and the Tracy-Widom Law

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Introduction

Theorem (Tracy-Widom Law)

As $n \to \infty$, the probability that all eigenvalues of an $n \times n$ GUE matrix are at most u is given by

$$\det(I - K_{Airy}^{[u,\infty)}) = \exp\left(-\int_u^\infty (\alpha - u) g^2(\alpha) d\alpha\right),$$

where g satisfies the Painlevé II equation

$$g'' = xg + 2g^3$$

and has asymptotics $g(x) \to \frac{\exp(-\frac{2}{3}x^{3/2})}{2\sqrt{\pi}x^{1/4}}$ as $x \to \infty$.

Gap Probabilities for GUE

- Let $\{\phi_n\}_{n\geq 0}$ be orthonormal functions given by $\phi_n(x) = \psi_n(x)e^{-\frac{1}{2}x^2}$, where $\{\psi_n\}_{n\geq 0}$ are the Hermite polynomials.
- Eigenvalues of a $N \times N$ GUE matrix have correlation functions

$$\rho(x_1, \ldots, x_k) = \det(K_N(x_i, x_j))_{i,j=1}^k$$

for the kernel

$$K_N(x, y) = \sum_{i=0}^{N-1} \phi_i(x)\phi_i(y).$$

- Gap probabilities are

$$\mathbb{P}(\text{no } x_i \text{ in } E) = \det(I - K_N^E),$$

where $K_N^E(x, y) = K_N(x, y) I_E$.

Airy Kernel

Theorem

Taking convergence in the sup-*norm, we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} K_n(\sqrt{2n} + \frac{x}{\sqrt{2}n^{1/6}}, \sqrt{2n} + \frac{y}{\sqrt{2}n^{1/6}}) = K_{Airy}(x, y),$$

where the Airy kernel is given by

$$K_{Airy}(x, y) = \int_0^\infty \mathrm{Ai}(x + t)\,\mathrm{Ai}(y + t)dt = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y}$$

and the Airy function is given by

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{u^3}{3} + xu\right) du$$

and satisfies the differential equation $\mathrm{Ai}''(x) - x\,\mathrm{Ai}(x) = 0.$

Background on KP

- Consider a pseudo-differential operator

$$L(t) = \partial_x + u_1(x,t)\partial_x^{-1} + u_2(x,t)\partial_x^{-2} + \cdots$$

Want two conditions:

- Continuous spectrum with wave function $\Psi(x,t,z)$:

$$L(t) \cdot \Psi(x,t,z) = z\Psi(x,t,z).$$

- Evolution of the wave function $\Psi(x,t,z)$:

$$\partial_i \Psi(x,t,z) = L_+^i \Psi(x,t,z),$$

where L_+^i is the purely differential part of L^i and $\partial_i = \frac{\partial}{\partial t_i}$.

KP hierarchy = conditions for this to be consistent

Background on KP (II)

- *Sato Grassmannian* Ω = Subspaces $W \subset \mathbb{C}[[z, z^{-1}]]$ with $\pi : W \xrightarrow{\sim} \mathbb{C}[[z]]$.
- Points in Ω correspond to τ functions:

$$\tau(t) = \det\left(\pi : e^{-\sum_{j\geq 1} t_j z^j} \cdot W \to \mathbb{C}[[z]]\right),$$

Theorem

Solutions $\{u_i\}$ to KP with wave function $\Psi(t, z)$ correspond to τ functions:

$$\Psi(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} e^{\sum_{i\geq 1} t_i z^i},$$

where $[z^{-1}] = \left(z^{-1}, \frac{z^{-2}}{2}, \frac{z^{-3}}{3}, \ldots\right)$.

- Can characterize all $\tau(t)$ by a bilinear relation.

KP and the Airy Kernel

Example of KP:

- Take $L(t)$ so that L^2 is purely differential and $L(0)^2 = \partial_x^2 - x$.
- Match wave function and Airy function:

$$\Psi(x,0,z) = 2\sqrt{\pi z}\,\mathrm{Ai}(x+z^2) = e^{xz+\frac{2}{3}z^3}\Big(1+o(1)\Big).$$

- $L(0)^2 \cdot \Psi(x,0,z) = (x+z^2)\Psi(x,0,z) - x\Psi(x,0,z) = z^2\Psi(x,0,z)$

Extend $\Psi(x,0,z)$ to all times:

- Consider asymptotics:

$$\Psi(x,0,z) = e^{xz+\frac{2}{3}z^3}(1+o(1)),$$

- Define subspace in Ω:

$$W = \mathrm{span}_{i\geq 0}\left\{\partial_1^i \Psi(x,0,z)\right\}.$$

KP and the Airy kernel (II)

Recall $W = \text{span}_{i \geq 0} \left\{ \partial_1^i \Psi(x, 0, z) \right\} \in \Omega$:

- Take τ function associated to W (Kontsevich integral):

$$\tau_{\text{Airy}}(t) = \lim_{N \to \infty} \frac{\int \exp\left(-\operatorname{Tr}(\frac{1}{3}X^3 + X^2 Z)\right) dX}{\int \exp\left(-\operatorname{Tr}(X^2 Z)\right) dX},$$

where X is drawn from $N \times N$ GUE and $Z = \text{diag}(z_n)$ with

$$t_n = -\frac{1}{n} \sum_i z_i^{-n} + \frac{2}{3} \delta_{n,3}.$$

- By Theorem, get $\Psi(x, t, z)$ corresponding to $\tau_{\text{Airy}}(t)$
- Check (abstractly) that $\Psi(x, 0, z) = 2\sqrt{\pi z}\, \text{Ai}(x + z^2)$.

Vertex operator and Airy kernel

- KP vertex operator:

$$X(t,y,z) := \frac{1}{z-y} \exp\left(\sum_{i\geq 1}(z^i - y^i)t_i\right) \exp\left(\sum_{i\geq 1} \frac{y^{-i} - z^{-i}}{i}\partial_i\right),$$

- For kernels of the form

$$K^E(t,y,z) = \int_E \Psi(x,t,y)\Psi^*(x,t,-z)dx,$$

can write

$$K^E(t,y,z) = \frac{X(t,y,z)\tau(t)}{\tau(t)}.$$

Fredholm determinants

Theorem

The Fredholm determinant of K^E is given by:

$$\det(I - \lambda K^E) = \frac{1}{\tau(t)} \exp\left(-\lambda \int_E X(t, z, -z) dz\right) \tau(t).$$

Proof idea: Consider discrete analogue and take limit.

- "$X(t, y, z)^2 = 0$", so $\exp(aX(t, y, z)) = 1 + aX(t, y, z)$, giving

$$\frac{1}{\tau(t)} \exp\left(\sum_i a_i X(t, z_i, -z_i)\right) \tau(t) = \frac{1}{\tau(t)} \prod_i (1 + a_i X(t, z_i, -z_i)) \tau(t).$$

- Expand and use identity on product of vertex operators to get

$$\det(I + a_j \int \Psi(x, t, z_i) \Psi^*(x, t, -z_i) dx).$$

Virasoro constraints

Consider the expansion:

$$X(t,y,z) = \frac{1}{z-y} \sum_{k=0}^{\infty} \frac{(z-y)^k}{k!} \sum_{l=-\infty}^{\infty} y^{-l-k} W_l^{(k)}.$$

- $W_l^{(1)}$ = realization of Heisenberg algebra
- $W_l^{(2)}$ = realization of Virasoro algebra

Commutation relations among $X(t,y,z)$ and $X(t,y',z')$ give:

$$\left[\frac{1}{2} W_l^{(2)}, X(t,z,-z)\right] = \partial_z \Big(z^{l+1} X(t,z,-z) \Big).$$

Integration by parts:

$$\left[\frac{1}{2} W_l^{(2)}, \int_a^b X(t,z,-z) dz\right] = b^{l+1} X(t,b,-b) - a^{l+1} X(t,a,-a).$$

Virasoro constraints (II)

Recall:

$$\left[\frac{1}{2}W_l^{(2)}, \int_a^b X(t,z,-z)dz\right] = b^{l+1}X(t,b,-b) - a^{l+1}X(t,a,-a)$$

and

$$\det(I - \lambda K^{[a,b]}) = \frac{1}{\tau(t)} \exp\left(-\lambda \int_a^b X(t,z,-z)dz\right)\tau(t).$$

If had $W_l^{(2)} \cdot \tau(t) = c_l \tau(t)$ (obtained from bilinear relations on τ), then combining gives

$$\left(b^{l+1}\partial_b + a^{l+1}\partial_a - \frac{1}{2}W_l^{(2)} + \frac{1}{2}c_l\right)\exp\left(-\lambda \int_E X(t,z,-z)dz\right)\tau(t) = 0,$$

so we see that

$$\left(b^{l+1}\partial_b + a^{l+1}\partial_a - \frac{1}{2}W_l^{(2)} + \frac{1}{2}c_l\right)\tau(t)\ker(I - \lambda K^{[a,b]}) = 0.$$

Virasoro constraints (III)

- Recall $\left(b^{l+1}\partial_b + a^{l+1}\partial_a - \frac{1}{2}W_l^{(2)} + \frac{1}{2}c_l\right)\tau(t)\ker(I - \lambda K^{[a,b]}) = 0$.
- General KP theory:

$$\widetilde{\tau}(t) := \tau(t)\ker(I - \lambda K^{[a,b]}) = \exp\left(-\lambda \int_E X(t, z, -z)dz\right)\tau(t)$$

  is a τ function.
- Use bilinear relations on $\widetilde{\tau}(t)$ in terms of t_i to get relations on $\widetilde{\tau}(t)$ in terms of ∂_a and ∂_b!
- Obtain constraints of form

$$P(a, b, \partial_a, \partial_b)\log(\tau(t)\ker(I - \lambda K^{[a,b]})) = 0.$$

- Can remove $\tau(t)$ because differential is independent of t.

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