VIRASORO CONSTRAINTS AND THE TRACY-WIDOM LAW

YI SUN

Abstract. We give a proof of the Tracy-Widom law from the perspective of M. Adler, T. Shiota, and P. van Moerbeke. Our exposition will use the identification of the Airy kernel as a certain kernel related to τ functions of the KP integrable system to obtain Virasoro constraints on its Fredholm determinant. These constraints will then give the standard Painlevé form of the Tracy-Widom law. A key technical ingredient in this treatment will be the KP vertex operator and its action on τ functions.

1. Introduction

1.1. **Motivation.** In [TW94], C. Tracy and H. Widom showed that the Fredholm determinant $\det(I-K^E_{\text{Airy}})$ satisfied a system of non-linear PDE's, which, when specialized to the case where $E=[u,\infty)$ is a semi-infinite interval, yields the following result, which is now known as the Tracy-Widom law.

Theorem 1.1 (Tracy-Widom Law)**.** *As $n\to\infty$, the probability that all eigenvalues of an $n\times n$ GUE matrix are at most u is given by*

$$\det(I-K^{[u,\infty)}_{Airy}) = \exp\left(-\int_u^\infty (\alpha-u)g^2(\alpha)d\alpha\right), \tag{1}$$

where g satisfies the Painlevé II equation

$$g'' = xg + 2g^3$$

and has asymptotics $g(x)\to \frac{\exp(-\frac{2}{3}x^{3/2})}{2\sqrt{\pi}x^{1/4}}$ as $x\to\infty$.

The original proof of Tracy and Widom was based on some analytic manipulation of Fredholm determinants which appear somewhat difficult to reconstruct. In the years afterwards, however, a new proof was developed in the work of M. Adler, T. Shiota, and P. van Moerbeke using ideas from integrable systems and infinite dimensional Lie algebras. The key idea is to reinterpret the Airy kernel K_{Airy} which appears in (1) as a kernel which emerges naturally from the study of τ functions for the KP hierarchy, an infinite family of partial differential equations which appears in the study of integrable systems. The KP hierarchy is related quite closely to the Virasoro algebra, and this relation allows one to explicitly characterize the action of a particular realization of the Virasoro algebra on the Fredholm determinants of interest. Interpreted in the situation of the Tracy-Widom law, this action is via certain differential operators, which when specialized yield exactly the desired Painlevé II characterization.

In this paper, our primary goal will be to give a high level exposition of this approach. While its consequences are much more general than the Tracy-Widom law and it can be applied more widely, we will restrict our discussion to this case for clarity. Our story will involve technical aspects of several areas of mathematics which are traditionally rather far from the theory of random matrices; in particular, it will hinge critically on the KP integrable system and the corresponding vertex operators. For these aspects, we will omit most proofs, choosing instead to give illustrative examples of the key concepts and refer the reader to more comprehensive sources for a more complete exposition.

1.2. **Organization and references.** We now detail the organization of the remainder of this paper. In Section 2, we introduce the theory of the KP integrable system, its τ and wave functions, and its vertex operators. These objects will be fundamental in these results, but we confine ourselves to giving the definitions and summarizing the most important results. In Section 3, we identify a class of integral kernels which fit nicely into the KP framework and explain how to obtain so-called Virasoro constraints on the corresponding Fredholm determinants. In Section 4, we conclude by applying this general theory to the give

Date: May 15, 2012.
Final paper for 18.338, Spring 2012.

a proof of Theorem 1.1; for this, we identify the Airy kernel as a kernel of the aforementioned type and translate the Virasoro constraints from their original abstract formulation into explicit differential equations on the Fredholm determinant of its kernel.

None of the material in this paper is original, but rather based on the work of M. Adler, T. Shiota, and P. van Moerbeke. The primary references are their original series of papers [AvM95, AvM01, ASvM95, ASvM98]. For general background on integrable systems, we have also made reference to [BBT03] and [KR87] and the original papers [KM81, DKM81, DJKM81]. Finally, for details on the realization of the Airy kernel as a τ function, we used the original papers [Dij92] and [Kon92].

1.3. **Conventions and notation.** We will often write t to denote a sequence of countably many variables $t_1, t_2, t_3, \ldots$. For a formal variable z, we denote by $[z]$ the sequence of variables $[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots\right)$.

2. KP hierarchy and vertex operators

In this section we give a brief overview of the KP hierarchy, an integrable system defined in terms of the isospectral deformations of a pseudo-differential operator. Our main goal will be to express the connection between solutions of the entire hierarchy of differential equations to τ functions on Sato's Grassmannian. This relation is obtained through the use of vertex operators, which will play a larger role later on in our story.

2.1. **The KP hierarchy.** Consider the differential operator $L = \partial_x$. We would like to find so-called isospectral deformations of L with continuous spectrum. That is, we want to modify L to a pseudo-differential operator

$$L(t) = \partial_x + u_1(x,t)\partial_x^{-1} + u_2(x,t)\partial_x^{-2} + \cdots$$

of countably many variables $t_1, t_2, \ldots$ which satisfies two conditions. First, L has continuous spectrum with wave function $\Psi(x,t,z)$, meaning that

$$L(t) \cdot \Psi(x,t,z) = z\Psi(x,t,z). \tag{2}$$

Second, the evolution of the wave function $\Psi(x,t,z)$ along the time coordinates t_i is governed by

$$\partial_i \Psi(x,t,z) = L_+^i \Psi(x,t,z), \tag{3}$$

where T_+ denotes the purely differential part of a pseudo-differential operator T and ∂_i is used to mean $\frac{\partial}{\partial t_i}$.

A choice of deformation $L(t)$ defines a system of partial differential equations on $\Psi(x,t,z)$, which is not always consistent. The following proposition characterizes the conditions under which consistency occurs; such conditions will define the KP hierarchy.

Proposition 2.1. *The equations (2) and (3) define a consistent system of PDE's for $\Psi(x,t,z)$ if and only if for each i we have the compatibility relation*

$$\partial_i L(t) = [L^i(t)_+, L(t)]. \tag{4}$$

Observe that the compatibility relations (4) themselves define a system of partial differential equations for the functions $\{u_i\}$ in infinitely many variables $\{t_i\}$. This system will be the main object of interest in this section.

Definition 2.2. The *KP hierarchy* is the system of partial differential equations on the $u_i(x,t)$ defined by (4). A *solution* to the KP hierarchy is an infinite sequence of functions $u_i(x,t)$ satisfying (4).

In this paper, we will be concerned with the following special class of solutions to the KP hierarchy, where our pseudo-differential operator L is constrained to be the p^{th} root of an actual differential operator.

Definition 2.3. For an integer $p \geq 2$, the *p-reduction* of the KP hierarchy is the system of partial differential equations defined by (4) and the additional constraint $L(t)^p_- = 0$. In the case $p = 2$, the resulting system is known as the *KdV hierarchy*.

Example 2.4. We compute the first non-trivial PDE implied by (4). Suppose that

$$L = \partial_x + u_1\partial_x^{-1} + u_2\partial_x^{-2} + \cdots.$$

We then have

$$\begin{aligned}
L_+^1 &= \partial_x \\
L_+^2 &= \partial_x^2 + 2u_1 \\
L_+^3 &= \partial_x^3 + 3u_1\partial_x + 3(u_2 + \partial_x u_1).
\end{aligned}$$

Computing the relation (4) explicitly, we see that

$$\begin{aligned}
[L_+^1, L] &= \partial_x u_1 \partial_x^{-1} + \partial_x u_2 \partial_x^{-2} + o(\partial_x^{-2}) \\
[L_+^2, L] &= (\partial_{xx} u_1 + 2\partial_x u_2)\partial_x^{-1} + (\partial_{xx} u_2 + 2u_1\partial_x u_1)\partial_x^{-2} + o(\partial_x^{-2}) \\
[L_+^3, L] &= (\partial_{xxx} u_1 + 3\partial_{xx} u_2 + 6u_1\partial_x u_1)\partial_x^{-1} + o(\partial_x^{-1}).
\end{aligned}$$

We may therefore compute

$$\partial_{22} u_1 = \partial_2(\partial_{xx} u_1 + 2\partial_x u_2) = \partial_{xxxx} u_1 + 4\partial_{xxx} u_2 + 4(\partial_x u_1)^2 + 4u_1\partial_{xx} u_1$$

and

$$\begin{aligned}
\partial_1\Big(4\partial_3 u_1 - 12u_1\partial_1 u_1 - \partial_{111} u_1\Big) &= \partial_1\Big(3\partial_{xxx} u_1 + 12\partial_{xx} u_2 + 12u_1\partial_x u_1\Big) \\
&= 3\partial_{xxxx} u_1 + 12(\partial_x u_1)^2 + 12u_1\partial_{xx} u_1 + 12\partial_{xxx} u_2,
\end{aligned}$$

obtaining the relation

$$3\partial_{22} u_1 = \partial_1\Big(4\partial_3 u_1 - 12u_1\partial_1 u_1 - \partial_{111} u_1\Big). \tag{5}$$

If we suppose that L is a solution to the KdV hierarchy, we see that $L_+^2 = L^2$, so (4) implies that $\partial_2 L(t) = 0$. In this case, the previous equation (5) reduces to

$$4\partial_3 u_1 - 12u_1\partial_1 u_1 - \partial_{111} u_1 = 0. \tag{6}$$

Remark. The relations (5) and (6) are known as the *KP equation* and *KdV equation*, respectively, explaining the naming convention for these hierarchies. In fluid mechanics, they occur naturally in study of shallow water waves and are of independent interest because they admit so-called *soliton solutions*. We refer the reader to [BBT03] for more details.

Remark. Thus far, we have presented $\Psi(x, t, z)$ as a function of scalar variables x and z and a vector variable $t = (t_1, t_2, t_3, \ldots)$. In fact, by (3), we have $\partial_i \Psi = \partial_x \Psi$, meaning that every wave function Ψ under consideration will take the form

$$\Psi(x, t, z) = \Psi'(x + t, z),$$

for some function $\Psi'(t, z)$, where by abuse of notation we define

$$x + t := (x + t_1, t_2, t_3, \ldots).$$

It is clear that Ψ' and Ψ determine each other uniquely, so in what follows we will abuse notation to simply write $\Psi(x, t, z)$ or $\Psi(t, z)$. This reflects the fact that x and t_1 play somewhat interchangeable roles.

By Proposition 2.1, every solution of (4) defines a corresponding function $\Psi(x, t, z)$, which is known as a *wave function* or *Baker-Akhiezer* function for the KP hierarchy. We may always write $\Psi(x, t, z)$ in the form

$$\Psi(x, t, z) = \Phi(x, t)e^{(x+t_1)z + \sum_{i\geq 2} t_i z^i}, \tag{7}$$

where $\Phi(x, t)$ takes the form

$$\Phi(x, t) = 1 + \sum_{i=1}^{\infty} w_i(x, t)\partial_x^{-i}.$$

This is always possible because $\partial_x^{-i} e^{(x+t_1)z + \sum_{j\geq 2} t_j z^j} = z^{-i} e^{(x+t_1)z + \sum_{j\geq 2} t_j z^j}$. Let us now define its formal adjoint to be

$$\Psi^*(x, t, z) = \Phi^*(x, t)e^{-(x+t_1)z - \sum_{i\geq 2} t_i z^i},$$

where Φ^* is the adjoint of Φ as a pseudo-differential operator. A key feature of the wave function is the following bilinear constraint.

Proposition 2.5. *Let $\Psi(t,z)$ be a wave function satisfying (2) and (3) for some pseudo-differential operator $L(t)$. Then for any t, t', we have*
$$\oint_{z=\infty} \Psi(t,z)\Psi(t',z)\frac{dz}{2\pi i} = 0,$$
where the contour integral is taken around a large circle at ∞.

Given a wave function $\Psi(x,t,z)$ satisfying the bilinear relation of Proposition 2.5, we may recover $L(t)$ as follows. With the choice of Φ made in (7), set
$$L(t) = \Phi(x,t)\partial_x \Phi(x,t)^{-1}.$$
This choice of $L(t)$ evidently satisfies (2), and (3) will follow from the bilinear relation.

2.2. **Sato's Grassmannian and τ functions for KP.** Perhaps miraculously, it is possible to characterize explicitly all solutions to the KP hierarchy (4). They will be parametrized by points (called τ-functions) on an infinite-dimensional geometric object known as Sato's semi-infinite Grassmannian. While this correspondence exists in general, we will restrict our attention from now on to 2-reduced solutions to the KP hierarchy, which are equivalent to solutions to the KdV hierarchy.

For a Baker-Akhiezer function $\Psi(t,z)$ associated to a 2-reduced solution to the KP hierarchy, we may vary t and consider the functions $\Psi_t := \Psi(t,z)$ on $\mathbb{C}^*$. Their linear span W_Ψ as we vary t is therefore a linear subspace of $\mathbb{C}[[z,z^{-1}]]$, the space of functions[1]. Choose an initial point $t = t^*$ and considering a Taylor series expansion of $\Psi(t,z)$ in the variables t_i about t^*. By the constraint (3), we may express $\partial_j \Psi(t^*,z)$ as a linear combination of $\partial_1^i \Psi(t^*,z)$ for all j, which implies that
$$W_\Psi = \mathrm{span}_{i\geq 0}\left\{\partial_1^i \Psi(t^*,z)\right\}.$$
By the representation of (7), observe that $\Psi(t^*,z)$ takes the form
$$\Psi(t^*,z) = (1+o(1))e^{\sum_{j\geq 1} t_j^* z^j},$$
so by induction we see that
$$\partial_1^i \Psi(t^*,z) = (z^i + o(z^i))e^{\sum_{j\geq 1} t_j^* z^j}.$$
In particular, this means that the restriction of the natural projection $\pi : \mathbb{C}[[z,z^{-1}]] \to \mathbb{C}[[z]]$ to W_ψ is an isomorphism. We are therefore in the following general situation, first proposed by Sato.

Definition 2.6. A linear subspace W of $\mathbb{C}[[z]]$ lies in Sato's *semi-infinite Grassmannian* Ω if and only if the projection $\pi : W \to \mathbb{C}[[z]]$ is an isomorphism. In this case, the *τ-function* of W is defined to be
$$\tau(t) = \det\left(\pi : e^{-\sum_{j\geq 1} t_j z^j}\cdot W \to \mathbb{C}[[z]]\right),$$
viewed as a function of infinitely many variables t_i. We will abuse notation to say that τ lies in Ω if it is the τ function of some subspace W in Ω.

Remark. We notice that τ is well-defined because the property of π restricting to an isomorphism on W is invariant under multiplication of W by $e^{-\sum_{j\geq 1} t_j z^j}$. Further, given τ, we may uniquely recover the original space W.

Viewed from this perspective, the mapping $\Psi \mapsto W_\Psi$ yields a map from solutions of the KP hierarchy to points of Ω, and hence to τ functions in Ω. To study the wave functions Ψ, then, we wish to characterize the τ functions. In analogy to the situation for finite-dimensional Grassmannians, this characterization will be in terms of Plücker relations. For this, we first formalize the setting where the τ functions will live.

Definition 2.7. The *Fock space* F is defined to be the ring $\mathbb{C}[t_1, t_2, \ldots]$ of polynomials in countably many variables.

Proposition 2.8 ([KR87, Proposition 7.3])**.** *Let τ be an element in $F = \mathbb{C}[t_1, t_2, \ldots]$. Then τ is a τ function for Ω if and only if the constant coefficient with respect to u of the power series*
$$u\exp\left(-\sum_{j\geq 1} 2u^j y_j\right)\exp\left(\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\right)\tau(t-y)\tau(t+y)$$

[1]We should specify certain convergence conditions, but will suppress these here.

vanishes as a power series in t and y.

As was the case with $\Psi(t,z)$, though $\tau \in \Omega$ is defined as a function of $t_1, t_2, \ldots$ alone, we will usually invoke τ with an argument of the form $\tau(x+t)$. We then have the following remarkable theorem, which characterizes all solutions to (4).

Theorem 2.9 ([KM81, Theorem 1]). *Solutions $\{u_i\}$ to (4) with wave function $\Psi(t,z)$ correspond to τ functions in Ω via the relation*

$$\Psi(t,z) = \frac{\tau(t-[z^{-1}])}{\tau(t)} e^{\sum_{i\geq 1} t_i z^i}, \tag{8}$$

where we recall the notation $[z^{-1}] = \left(z^{-1}, \frac{z^{-2}}{2}, \frac{z^{-3}}{3}, \ldots\right)$.

According to Theorem 2.9, solutions to (4) are specified abstractly by τ functions. Our purposes will require us to transition between these τ functions, the corresponding Baker-Akhiezer functions $\Psi(t,z)$, and the solutions $u_i(x,t)$ to the KP hierarchy. We now detail some directions of this process more explicitly. First, we illustrate how to find solutions of the classical KP equation in this way.

Example 2.10. We will compute $u_1(x,t)$ in terms of τ. Taking the constraint (3) for $i=2$, we obtain the identity

$$\partial_2 \Psi(x,t,z) = (\partial_x^2 + 2u_1(x,t))\Psi(x,t,z),$$

which we view as an equality of formal power series in $z, t_1, t_2, t_3, \ldots$. Taking a power series expansion of (8), we find that

$$[z^0]\Psi = \frac{1}{\tau}\left(1 - \partial_1 \tau t_1 + (\partial_{11}\tau - \frac{1}{2}\partial_2\tau)(\frac{1}{2}t_1^2 + t_2)\right),$$

which shows that

$$2u_1 = [\text{const}]2u_1\Psi = [\text{const}](\partial_2\Psi - \partial_{11}\Psi) = \left(\frac{\partial_{11}\tau - \frac{1}{2}\partial_2\tau}{\tau}\right) - \left(-2\partial_{11}\log\tau + \frac{\partial_{11}\tau - \frac{1}{2}\partial_2\tau}{\tau}\right) = 2\partial_{11}\log\tau,$$

yielding the expression $u_1 = \partial_{11}\log\tau$ for the solution of the classical KP equation (5) in terms of the τ function.

More importantly, we would like to find a way to invert (8) and express τ in terms of Ψ. In principle, our definition of W_Ψ above has already given such an expression; however, we can make it more explicit by placing it in the following determinantal form.

Proposition 2.11 ([Kon92, Lemma 4.2]). *Let $W = \operatorname{span}\{w_1, w_2, \ldots\}$ in Ω have basis $\{w_i\}$ with $w_i = z^i + o(z^i)$. Then the τ function of W is*

$$\tau(t) = \lim_{n\to\infty} \frac{\det(w_i(z_j))_{i,j=1}^n}{\det(z_j^{-1})_{i,j=1}^n},$$

where z_k are the Miwa coordinates defined by $t_k = \frac{1}{k}\sum_{i=1}^n z_i^k$.

2.3. **Vertex operators for KP.** Ultimately, we would like to obtain identities satisfied by functions built from τ functions. Such identities will arise from deformations of the bilinear relation of Proposition 2.5; however, some of these quite intricate identities will require the use of the theory of vertex operators. We give a brief summary of this theory as it applies to our situation in this subsection.

Definition 2.12. The *KP vertex operator* is defined as

$$X(t,y,z) := \frac{1}{z-y}\exp\left(\sum_{i\geq 1}(z^i - y^i)t_i\right)\exp\left(\sum_{i\geq 1}\frac{y^{-i}-z^{-i}}{i}\partial_i\right), \tag{9}$$

considered as a formal power series of operators on Fock space F.

Careful use of Proposition 2.5 will give the following technical result on $X(t,y,z)$, which will be key for our purposes.

Proposition 2.13. *Let $\Psi(x,t,z)$ and $\Psi^*(x,t,z)$ be the wave function and adjoint wave function corresponding to a τ function $\tau(t)$. Then we have the following:*

(1) The action of $X(y,z)$ on $\tau(t)$ is given by

$$\frac{X(t,y,z)\tau(t)}{\tau(t)} = \int \Psi(x,t,y)\Psi^*(x,t,z)dx; \tag{10}$$

(2) The composition of actions of different $X(y_i,z_i)$ on $\tau(t)$ is given by a determinant

$$\frac{X(t,y_1,z_1)\cdots X(t,y_n,z_n)\tau(t)}{\tau(t)} = \det\left(\int \Psi(x,t,y_i)\Psi^*(x,t,z_j)dx\right)_{ij}. \tag{11}$$

We note now that it is a very special property of vertex operators that they send τ functions to τ functions. In fact, this may in some ways be viewed as a defining property of the vertex operators. More specifically, we have the following proposition.

Proposition 2.14 ([KR87, Corollary 7.2])**.** *For any τ function $\tau(t)$ of the KP hierarchy and any $\lambda \in \mathbb{R}$, the function*

$$e^{\lambda X(t,y,z)}\tau(t)$$

is a τ function of the KP hierarchy for any y,z.

We would like now to obtain generators of the Virasoro algebra as coefficients of a certain expansion of the vertex operator $X(t,y,z)$. Consider the expansion

$$X(t,y,z) := \frac{1}{z-y}\sum_{k=0}^{\infty}\frac{1}{k!}(z-y)^k\sum_{l=-\infty}^{\infty} y^l W_l^{(k)}, \tag{12}$$

where $W_l^{(k)}$ are certain operators on Fock space F which act locally finitely on any element of F. We will focus on the cases where $0 \leq k \leq 2$, where we obtain

$$W_l^{(0)} = \delta_{l,0} \tag{13}$$

$$W_l^{(1)} = \begin{cases} \partial_l & l > 0 \\ -lt_{-l} & l < 0 \\ 0 & l = 0 \end{cases} \tag{14}$$

$$W_l^{(2)} = \sum_{i+j=l} :W_i^{(1)}W_j^{(1)}: -(l+1)W_l^{(1)}, \tag{15}$$

where $:W_i^{(1)}W_j^{(1)}:$ denotes the *normal ordering*[2]. Using the explicit expressions for $W_l^{(1)}$ and $W_l^{(2)}$, we may check that these operators form representations of the Heisenberg and Virasoro algebras.

Proposition 2.15. *The operators $\{W_l^{(1)}\}$ and $\{W_l^{(2)}\}$ form representations of the Heisenberg and Virasoro algebras, respectively, on Fock space F.*

The most important feature of the operators $W_l^{(2)}$ will be the following *vertex Lax relation*. It may be obtained from the explicit formula for $W_l^{(2)}$ by a direct calculation, but we emphasize it here because of the prominent role it will play later.

Proposition 2.16. *The commutation relation*

$$\left[\frac{1}{2}W_l^{(2)}, X(t,z,-z)\right] = \partial_z\left(z^{l+1}X(t,z,-z)\right) \tag{16}$$

holds for all t and z.

[2]For a differential operator D on F given as the product of monomials and differentials ∂_i, the normal ordering of D is given by rearranging the product so that all differentials are on the right. For instance, we have $:\partial_1 t_1 := t_1\partial_1 =: t_1\partial_1:$.

3. Virasoro constraints on Fredholm determinants

In this section, we apply the vertex operator machinery developed in the previous section to obtain the main technical results, which are differential relations on certain Fredholm determinants of kernels constructed from Baker-Akheizer functions. More explicitly, let $\Psi(x, t, z)$ be a Baker-Akhiezer function for the 2-reduced KP hierarchy, and let $E \subset \mathbb{R}$ be a domain. Our goal will be to study kernels of the form

$$K^E(t, y, z) = \int_E \Psi(x, t, y)\Psi^*(x, t, -z)dx, \tag{17}$$

which we expect to have good properties due to Proposition 2.13.

3.1. Fredholm determinants and τ functions.

In particular, we are interested in studying the Fredholm determinants $\det(I - \lambda K^E)$ of kernels of the form (17). Our main technique will be to use (10) to evaluate these determinants in terms of vertex operators. In particular, we have the following consequence of the vertex operator relations.

Proposition 3.1. *The Fredholm determinant of K^E is given by*

$$\det(I - \lambda K^E) = \frac{1}{\tau(t)} \exp\left(-\lambda \int_E X(t, z, -z)dz\right)\tau(t).$$

Proof. We may consider the defining expansion and apply part 2 of Proposition 2.13 to obtain

$$\begin{aligned}
\det(I - \lambda K^E)\tau(t) &= \sum_{k=0}^{\infty}(-\lambda)^k \int_E \cdots \int_E \det(K(x_i, x_j))_{ij}\tau(t)dx_1 \cdots dx_k \\
&= \sum_{k=0}^{\infty}(-\lambda)^k \int_E \cdots \int_E X(t, x_1, -x_1) \cdots X(t, x_k, -x_k)dx_1 \cdots dx_k\tau(t) \\
&= \sum_{k=0}^{\infty}(-\lambda)^k \left(\int_E X(t, z, -z)dz\right)^k \tau(t) \\
&= \exp\left(-\lambda \int_E X(t, z, -z)dz\right)\tau(t). \qquad \square
\end{aligned}$$

This expression allows us to realize $\tau(t)\det(I-\lambda K^E)$ as an actual τ function for the KP hierarchy. Indeed, this will follow from the characterization of vertex operators by Proposition 2.14 as follows.

Corollary 3.2. *The function $\det(I - \lambda K^E)\tau(t)$ is a τ function for the KP hierarchy.*

Proof. We consider the case where E is an interval, as the general case evidently follows from this. By Proposition 3.1, we have

$$\begin{aligned}
\det(I - \lambda K^E)\tau(t) &= \exp\left(-\lambda \int_E X(t, z, -z)dz\right)\tau(t) \\
&= \lim_{n\to\infty} \prod_{i=1}^{n} \exp\left(-\frac{\lambda}{n} X(t, z_i, -z_i)\right)\tau(t),
\end{aligned}$$

where $\{z_i\}$ form a partition of the interval E. By Proposition 2.14, we have thus expressed $\det(I-\lambda K^E)\tau(t)$ as a limit of τ functions for the KP hierarchy. Now, because the conditions that Proposition 2.8 imposes on τ are closed ones, we find that $\det(I - \lambda K^E)\tau(t)$ is itself a τ function, as desired. $\square$

3.2. Obtaining the Virasoro constraints.

Frequently, the τ functions we consider will admit so-called *Virasoro relations*, meaning that they are eigenfunctions for the operators $W_l^{(2)}$. For the remainder of this section, we will assume that we are given such relations of the form

$$W_l^{(2)}\tau(t) = c_l\tau(t). \tag{18}$$

Our goal will be to derive from the relations (18) similar relations on the Fredholm determinant $\det(I-\lambda K^E)$, known as *Virasoro constraints*.

Proposition 3.3. *For any $a \in \mathbb{R}$, $\det(I - \lambda K^{[a,\infty)})\tau(t)$ satisfies the relation*

$$\left(a^{l+1}\partial_a - \frac{1}{2}W_l^{(2)} + \frac{1}{2}c_l\right)\det(I - \lambda K^{[a,\infty)})\tau(t). \tag{19}$$

Proof. Using Proposition 2.16 and integration by parts, we may then compute

$$\begin{aligned}0 &= \int_a^\infty \left(\partial_z\left(z^{l+1}X(t,z,-z)\right) - \left[\frac{1}{2}W_l^{(2)}, X(t,z,-z)\right]\right)dz \\ &= -a^{l+1}X(t,a,-a) - \left[\frac{1}{2}W_l^{(2)}, \int_a^\infty X(t,z,-z)dz\right] = \left(a^{l+1}\partial_a - \left[\frac{1}{2}W_l^{(2)}, -\right]\right)\int_a^\infty X(t,z,-z)dz,\end{aligned}$$

which implies that

$$\left[\frac{1}{2}W_l^{(2)}, \int_a^\infty X(t,z,-z)dz\right] = a^{l+1}\partial_a \int_a^\infty X(t,z,-z)dz.$$

Thus, by considering power series expansions, we find that

$$\left[\frac{1}{2}W_l^{(2)}, \exp\left(-\lambda\int_a^\infty X(t,z,-z)dz\right)\right] = a^{l+1}\partial_a \exp\left(-\lambda\int_a^\infty X(t,z,-z)dz\right).$$

Therefore, by Proposition 3.1, we see that

$$\begin{aligned}\frac{1}{2}W_l^{(2)}\tau(t)\det(I - \lambda K^{[a,\infty)}) &= \left[\frac{1}{2}W_l^{(2)}, \exp\left(-\lambda\int_a^\infty X(t,z,-z)dz\right)\right]\tau(t) + \exp\left(-\lambda\int_a^\infty X(t,z,-z)dz\right)\cdot\frac{1}{2}W_l^{(2)}\tau(t) \\ &= \left(a^{l+1}\partial_a + \frac{1}{2}c_l\right)\det(I - \lambda K^{[a,\infty)})\tau(t),\end{aligned}$$

which simplifies to the desired relation (19). □

Remark. We may use Corollary 3.2 to connect the different relations given by (19). In particular, notice that the operators $W_l^{(2)}$ are differential operators on Fock space F, meaning they involve derivatives in the variables t_i. Because $\det(I - \lambda K^{[a,\infty)})\tau(t)$ is a τ function for the KP hierarchy, its t_i-derivatives satisfy certain bilinear relations, which we may translate to relations on $\left(\lambda a^{l+1} - \frac{1}{2}c_l\right)\det(I - \lambda K^{[a,\infty)})\tau(t)$, where now the operator we apply to $\det(I - \lambda K^E)\tau(t)$ is independent of t. Thus, we obtain constraints on the Fredholm determinant $\det(I - \lambda K^{[a,\infty)})$ alone. While such constraints may be very complicated in general, we will consider an explicit example which we hope will elucidate the general case.

4. Specializing to the Tracy-Widom law

In this section, we apply the general machinery we have developed so far to give a conceptual proof of Theorem 1.1, the Tracy-Widom law. The proof follows the standard approach of expressing the probability distribution of the maximal eigenvalue of a GUE matrix as the Fredholm determinant of a certain Airy kernel. We then identify this kernel as one which may be obtained from the general formalism of τ functions, allowing us to obtain our desired result by specializing the Virasoro constraints of (19).

4.1. **Soft scaling limits for the GUE.** We begin by sketching the standard reduction to the computation of the Fredholm determinant of the Airy kernel. Let $\{\phi_n\}_{n\geq 0}$ be the system of orthonormal functions given by $\phi_n(x) = \psi_n(x)e^{-\frac{1}{2}x^2}$, where $\{\psi_n\}_{n\geq 0}$ are the Hermite polynomials. Then the eigenvalues of a $N \times N$ GUE matrix have correlation functions

$$\rho(x_1, \ldots, x_k) = \det(K_N(x_i, x_j))_{i,j=1}^k$$

for the kernel

$$K_N(x,y) = \sum_{i=0}^{N-1}\phi_i(x)\phi_i(y).$$

Further, for a union of open intervals $E \subset \mathbb{R}$, we may use this kernel to express the gap probabilities

$$\mathbb{P}(\text{no } x_i \text{ in } E) = \det(I - K_N^E),$$

where $K_N^E(x,y) = K_N(x,y)I_E$, and where the determinant is a Fredholm determinant. The scaling limit of this kernel at the soft edge is classically known to result in the Airy kernel via the following result.

Theorem 4.1. *Taking convergence in the* sup*-norm, we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} K_n(\sqrt{2n} + \frac{x}{\sqrt{2}n^{1/6}}, \sqrt{2n} + \frac{y}{\sqrt{2}n^{1/6}}) = K_{Airy}(x, y),$$

where the Airy kernel is given by

$$K_{Airy}(x, y) = \int_0^\infty \operatorname{Ai}(x+t)\operatorname{Ai}(y+t)dt = \frac{\operatorname{Ai}(x)\operatorname{Ai}'(y) - \operatorname{Ai}'(x)\operatorname{Ai}(y)}{x-y}$$

and the Airy function is given by

$$\operatorname{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{u^3}{3} + xu\right) du$$

and satisfies the differential equation $\operatorname{Ai}''(x) - x\operatorname{Ai}(x) = 0$.

It is known that the Fredholm determinant of a kernel is continuous with respect to the sup-norm, which means that Theorem 4.1 implies that

$$\lim_{n\to\infty} \det(I - K_n^E) = \det(I - K_{\text{Airy}}^E),$$

so to understand the scaling limit, it suffices to understand this single Fredholm determinant.

4.2. **Realizing the Airy kernel.** The next step will be to realize K_{Airy} in the form (17). For this, set $t^* = (0, 0, 2/3, 0, \ldots)$. We would like to construct a τ function so that the corresponding wave function Ψ satisfies

$$\Psi(x, t^*, z) = 2\sqrt{\pi z}\operatorname{Ai}(x + z^2).$$

Define $\Psi_0(x, z) := 2\sqrt{\pi z}\operatorname{Ai}(x + z^2)$. We may verify that Ψ_0 has asymptotic expansion

$$\Psi_0(x, z) = e^{xz + \frac{2}{3}z^3}(1 + o(1)), \tag{20}$$

which implies the expansions

$$\partial_x^i \Psi_0(x, z) = e^{xz + \frac{2}{3}z^3}(z^i + o(z^i))$$

for its derivatives. We conclude that the subspace

$$W_0 := \operatorname{span}_{i\geq 0}\{\partial_x^i \Psi_0(0, z)\} \subset \mathbb{C}[[z, z^{-1}]]$$

lies in Ω. Therefore, W_0 corresponds to some τ function $\tau(t)$, which by Theorem 2.9 corresponds to a solution of the KP hierarchy with wave function $\Psi(t, z)$. Our objective is then to check that $\Psi(x, t^*, z) = \Psi_0(x, z)$. This will follow from the following general construction.

Proposition 4.2. *Let W be a subspace in the semi-infinite Grassmannian. Then, we have the following:*

(1) for each value of the variables $\{t_i\}$, $e^{-\sum_{i\geq 1} t_i z^i} \cdot W$ contains a unique element of the form

$$\eta(t, z) = 1 + \sum_{i\geq 1} a_i z^{-i};$$

(2) the wave function corresponding to W is given by

$$\Psi(t, z) = \eta(t, z) \cdot e^{\sum_{i\geq 1} t_i z^i}.$$

Corollary 4.3. *The function $\Psi_0(x, z)$ is the specialization of some Baker-Akhiezer function $\Psi(x, t^*, z)$ for the KP hierarchy.*

Proof. By Proposition 4.2, it suffices to check that

$$\eta_0(t^*, z) := e^{-\sum_{i\geq 1} t_i^* z^i} \cdot \Psi_0(x, z)$$

is of the form $1 + \sum_{i\geq 1} a_i z^{-i}$. This follows from our earlier asymptotic expression (20). □

Remark. Proposition 4.2 provides a general technique for realizing functions as Baker-Akhiezer functions for the KP hierarchy. In fact, for any t^* and any initial function $\Psi_0(x,z)$ normalized so that

$$e^{-\sum_{i\geq 1} t_i^* z^i} \cdot \Psi_0(x,z) = 1 + o(1),$$

taking the subspace $W \subset \mathbb{C}[[z]]$ spanned by Ψ_0 and some functions Ψ_i with $\Psi_i(x,z) = z^i + o(z^i)$ will yield a point in the semi-infinite Grassmannian and hence a solution to the KP hierarchy. Such a solution realizes $\Psi_0(x,z)$ in the form $\Psi(x,t^*,z)$ for some Baker-Akhiezer function and is referred to as taking the *KP flow* through Ψ_0.

We would like now to show that the kernel $K^E(t,y,z)$ we constructed has good properties; in particular, we wish to use it to shed some light on the Airy kernel $K^E_{\text{Airy}}(y,z)$.

Proposition 4.4. *We have an equality of Fredholm determinants*

$$\det\Big(I - \lambda K^{[a,\infty)}(t^*,y,z)\Big) = \det\Big(I - 2\pi\lambda K^{[a^2,\infty)}_{Airy}(y,z)\Big). \tag{21}$$

Proof. Computing using the defining property of Ψ, observe that

$$K^{[a^2,\infty)}(t^*,y,z) = \int_{a^2}^{\infty} \Psi(x,t^*,y)\Psi^*(x,t^*,z)dz = \int_{a^2}^{\infty} 4\pi\sqrt{yz}\,\mathrm{Ai}(x+y^2)\,\mathrm{Ai}(x+z^2)dx = 4\pi\sqrt{yz}K^{[a^2,\infty)}_{\text{Airy}}(y^2,z^2).$$

From the expansion of the Fredholm determinant, we may now obtain

$$\begin{aligned}
\det\Big(I - \lambda K^{[a,\infty)}(t^*,y,z)\Big) &= \sum_{k=0}^{\infty}(-\lambda)^k \int_a^{\infty}\cdots\int_a^{\infty} \det\Big(K(y_i,y_j)\Big)_{ij} dy_1\cdots dy_k \\
&= \sum_{k=0}^{\infty}(-\lambda)^k \int_a^{\infty}\cdots\int_a^{\infty} \det\Big(4\pi\sqrt{y_iy_j}K_{\text{Airy}}(y_i^2,y_j^2)\Big)_{ij} dy_1\cdots dy_k \\
&= \sum_{k=0}^{\infty}(-\lambda)^k \int_{a^2}^{\infty}\cdots\int_{a^2}^{\infty} \det\Big(2\pi K_{\text{Airy}}(y_i,y_j)\Big)_{ij} dy_1\cdots dy_k \\
&= \det\Big(I - 2\pi\lambda K^{[a^2,\infty)}_{\text{Airy}}(y,z)\Big),
\end{aligned}$$

where in the third equality we have used the fact that each term of the expanded determinant contains two factors of the from $\sqrt{y_i}$ for each i. □

We remark that our treatment of Corollary 4.3 avoids completely the specific form of the τ function associated to Ψ_0. In fact, this is known and may be computed by applying Proposition 2.11 to our given basis for W_0. The resulting function is the so-called *Kontsevich integral*

$$\tau_{\text{Airy}}(t) = \lim_{N\to\infty} \frac{\int \exp\left(-\operatorname{Tr}(\frac{1}{3}X^3 + X^2 Z)\right) dX}{\int \exp\left(-\operatorname{Tr}(X^2 Z)\right) dX}, \tag{22}$$

where the integrals are taken relative to X drawn from the $N\times N$ Gaussian unitary ensemble and where Z is a diagonal matrix with entries z_n so that

$$t_n = -\frac{1}{n}\sum_i z_i^{-n} + \frac{2}{3}\delta_{n,3}.$$

From the explicit form (22), it is possible to obtain Virasoro constraints on $\tau_{\text{Airy}}(t)$ based either on explicit calculations or general considerations from [AvM95]. We will not give details here, but the resulting constants for the constraints are given by

$$c_l = -\frac{1}{4}\delta_{l,0}. \tag{23}$$

4.3. **From Virasoro constraints to Painlevé II.** It remains for us to interpret the Virasoro constraints of Proposition 3.3 as differential equations on the Fredholm determinant $\det\left(I - \lambda K_{\mathrm{Airy}}^{[a,\infty)}\right)$ of our original Airy kernel in order to give a proof of Theorem 1.1. For this, we will pass through Proposition 4.4 and consider explicitly certain of the relations (19) using the equations of the KP hierarchy.

Proof of Theorem 1.1. Set $A = a^2$ and reparametrize (19) to obtain

$$\left(A^{l+1}\partial_A - \frac{1}{4}W_{2l}^{(2)} + \frac{1}{4}c_{2l}\right)\det(I - \lambda K_{\mathrm{Airy}}^{[A,\infty)})\tau(t)$$
$$= \frac{1}{2}\left(a^{2l+1}\partial_a - \frac{1}{2}W_{2l}^{(2)} + \frac{1}{2}c_{2l}\right)\det\left(I - \frac{\lambda}{2\pi}K^{[a,\infty)}\right)\tau(t) = 0.$$

In this case, by (23), the only non-zero value of c_{2l} is $c_0 = -\frac{1}{4}$. Therefore, taking $l = -1$ and $l = 0$ and writing

$$\widetilde{\tau}(t) = \det(I - \lambda K_{\mathrm{Airy}}^{[A,\infty)})\tau(t), \tag{24}$$

we obtain

$$\partial_A\widetilde{\tau}(t) = \frac{1}{4}W_{-2}^{(2)}\widetilde{\tau}(t) \tag{25}$$

$$A\partial_A\widetilde{\tau}(t) = \left(\frac{1}{4}W_0^{(2)} + \frac{1}{16}\right)\widetilde{\tau}(t), \tag{26}$$

where we recall that the operators $W_l^{(2)}$ are given explicitly by (15) and where the notation (24) is justified by Corollary 3.2. Computing these explicitly, we find that

$$\left(\frac{1}{4}W_{-2}^{(2)}\right)^i\widetilde{\tau}(t^*) = \partial_1^i\widetilde{\tau}(t^*)$$
$$\left(\frac{1}{4}W_0^{(2)}\right)\widetilde{\tau}(t^*) = \partial_3\widetilde{\tau}(t^*)$$
$$\left(\frac{1}{4}W_0^{(2)} + \frac{1}{16}\right)\left(\frac{1}{4}W_{-2}^{(2)}\right)\widetilde{\tau}(t^*) = \left(\partial_{13} + \frac{3}{2}\partial_1\right)\widetilde{\tau}(t^*),$$

where we note that the $\frac{3}{2}\partial_1$ term of the last equality comes from the fact that the composition $W_0^{(2)}W_{-2}^{(2)}$ is taken as the product of differential operators (so in particular the value $t = t^*$ can only be substituted after computing this composition). Now, because $\widetilde{\tau}$ is a τ function for the KdV hierarchy, the relations (6) and the fact that $u_1 = \partial_{11}\widetilde{\tau}$ imply that

$$\partial_{1111}\log\widetilde{\tau} - 4\partial_{13}\log\widetilde{\tau} + 6(\partial_{11}\widetilde{\tau})^2 = 0,$$

so we may substitute our previous computations to obtain

$$\left(\frac{1}{4}W_{-2}^{(2)}\right)^4\log\widetilde{\tau}(t^*) - 4\left(\left(\frac{1}{4}W_0^{(2)} + \frac{1}{16}\right)\left(\frac{1}{4}W_{-2}^{(2)}\right) - \frac{3}{8}W_{-2}^{(2)}\right)\log\widetilde{\tau}(t^*) + 6\left(\left(\frac{1}{4}W_{-2}^{(2)}\right)^2\log\widetilde{\tau}(t^*)\right)^2 = 0.$$

The Virasoro constraints (25) and (26) allow us to transform this into

$$(\partial_{AAAA} - 4A\partial_{AA} + 2\partial_A)\log\widetilde{\tau}(t^*) = (\partial_{AAAA} - 4\partial_A A\partial_A + 6\partial_A)\log\widetilde{\tau}(t^*) = -6\left(\partial_{AA}\log\widetilde{\tau}(t^*)\right)^2,$$

meaning that the expression $R = \partial_A\log\widetilde{\tau}(t^*)$ satisfies

$$\partial_{AAA}R - 4A\partial_A R + 2R + 6(\partial_A R)^2 = 0. \tag{27}$$

Applying the standard substitution $g^2 = -\partial_A R$ from the Hamiltonian theory of the Painlevé II equation, we obtain the relation

$$(3\partial_A g - g\partial_A)(\partial_{AA}g - 2g^3 - Ag) = 0,$$

which implies that $f = \partial_{AA}g - 2g^3 - Ag$ satisfies $3\partial_A g \cdot f = g \cdot \partial_A f$. Solving, this means that $f = cg$ for some constant c, hence g satisfies the equation

$$\partial_{AA}g = (2 + c)g^3 + Ag$$

for some constant c. Taking a bit more care with the computation of asymptotics for R and thus g will allow us to show that $c = 0$, yielding the standard form

$$\partial_{AA} g = 2g^3 + Ag,$$

of Painlevé II. Unwinding the definitions from our substitution, we see that

$$\partial_{AA} \log \det \left(I - \lambda K_{\text{Airy}}^{[A,\infty)}\right) \tau(t^*) = -g(A)^2,$$

so we may integrate twice and set $\lambda = 1$ to see that

$$\log \det \left(I - K_{\text{Airy}}^{[A,\infty)}\right) \tau(t^*) = \int_A^\infty \int_v^\infty g^2(u) du dv = \int_A^\infty \int_A^u g(u)^2 dv du = \int_A^\infty (u - A) g(u)^2 du.$$

Observing that $\tau(t^*) = 1$ for the Kontsevich integral (22), this implies finally that

$$\det \left(I - K_{\text{Airy}}^{[A,\infty)}\right) = \exp\left(-\int_A^\infty (A - u) g(u)^2 du\right),$$

which is the desired result (1). □

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