Non-coherent MIMO Communications

Mina Karzand
Research Laboratory of Electronics (RLE)
MIT

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Chapter 1

Introduction to Wireless Communications

1.1 The Communication Systems Mathematical Model

The communication channels model in general is consisted of three main components.

- $x[t] \in \mathcal{X}$ for $t = 1, 2, \cdots, T$: which is the sequence of input signals to the channel taking values in the alphabet $\mathcal{X}$. The message that is to be transmitted is conveyed in this sequence. There is also some redundancy in the representation of message in sequence $x[t]$ so that the communication would be robust towards the noise and the message can be decoded at the receiver reliably (i.e., with arbitrary small probability of error).

- $y[t] \in \mathcal{Y}$ for $t = 1, 2, \cdots, T$: which is the sequence of output signals from the channel taking values in the alphabet $\mathcal{Y}$. This sequence would probably be different from the input signal due to the random behavior of channel. This difference might cause difficulty in decoding the message.

- $P(y[t]|x[t])$ for $x[t] \in \mathcal{X}$ and $y[t] \in \mathcal{Y}$: The transition probability $P(y[t]|x[t])$ is the stochastic model of the channel which demonstrate the distribution of output signal for each time instant t for a given input $x[t]$.

There are two fundamental question that arises in design and analysis of communication systems:

- *What is the **capacity** of the given communication channel?* The capacity of the channel is defined to be the maximum amount of information that can be transmitted reliably over a given channel per use of the channel. Meaning that if B bits of information can be transmitted in a block of length T (in T times use of communication channel), the rate is defined to be B/T. The capacity of the channel is a measure of the quality of channel, meaning that if the channel is imposing too much randomness over the output for a given input, the recovery of the message gets more difficult and more redundancy should be added to the input sequence to make reliable recovery of the message feasible, the the capacity is smaller.

- *How should be design the system to achieve the maximum rate?* In other words, what is the optimal encoding and decoding scheme to so that the redundancy would help us recover the message reliably.

1.2 Point to Point Wireless Channel

In wireless communications, the communication channel is the open air and the space between the transmitter and the receiver. Attenuation, multi-path fading and noise are the main phenomena that affect the transmission of electromagnetic waves representing the input signals.

The stochastic behavior of the channel is modeled as the following.

$$\mathbf{y}[t] = \mathrm{SNR}\mathbf{h}[t]\mathbf{x}[t] + \mathbf{w}[t], \qquad \text{for } t = 1, \cdots, T$$

In this model the output signal at each time is a random function of the input signal. This function has three parameters that depend on the properties of the channel

- SNR: Signal to noise ratio in the channel measures the average ratio of the received power from the transmitted signal versus the noise in the system. This parameter is deterministic and quantifies the quality of the channel. Signal attenuation, the communication environment and the distance between transmitter and receiver are among the parameters that determine the value of SNR.

- $\mathbf{h}[t]$ Channel fading coefficient is the parameter that conveys the randomness of channel mainly due to multi-path fading. Using central limit theorem, this is usually modeled as a random complex number with Gaussian distribution, mean zero and unit variance.

- $\mathbf{w}[t]$ representing the additive noise of the system is modeled as a gaussian distributed random complex number with mean zero and unit variance.

There are several limiting factors in wireless communication looking at the model above. *Input power constraint*: which constraints the average power transmitted from the transmitter at a block of time. The transmitted signal should satisfy the inequality for some given P:

$$\frac{1}{T}\sum_{t=1}^{T} ||x[t]||^2 < P$$

Noise is another limiting factor in communication which distorts the received signal in a random manner.

Fading is the parameter that potentially can change the amplitude and phase of the transmitted signal in a random manner. The coding scheme should be resilient to this kind of randomness in the received signal. *Deep fading* is the phenomenon that the coefficient $h[t]$ has a very small magnitude and imposes a very small amplitude on the received signal compared to the noise. In this case, we say that the signal is drowned in noise and signal recovery is almost impossible.

1.2.1 Coherent Wireless Communication

In coherent wireless communication, the receiver has the full knowledge about the fading coefficients and the parameter $\mathbf{h}[t]$ is a random number which is known at the receiver and the transmitter. The main problem in coherent communication is distributing total power budget in a block of length T in a way that input power constraint is satisfied and at the same time the maximum amount of information is transmitted. Intuitively, the smart strategy is to allocate more power to the time slots that have smaller $|h[t]|$ so that even though the fading is strong, (i.e.,$|h[t]|^2 << 1$), the input power at that time is large enough (i.e., $|x[t]|^2 >> 1$) such that the received power, $|h[t]x[t]|^2$, would be sufficient to perform the decoding reliably.

1.2.2 Non-Coherent Wireless Communication

In non-coherent wireless communications, the receiver (neither the transmitter) are not aware of the actual realization of the channel coefficients. The encoding and decoding should be performed by using only the statistics of the coefficients.

1.2.3 Flat Fading

Flat fading is referred to the model in which the changes in channel are so slow that the channel coefficients are assumed to be constant over each block of communications. i.e., $h[t] = h$ for $t = 1, 2, \cdots, T$.

1.3 Single-input and Multiple-output (SIMO) Communications.

In SIMO communications, there are a single transmitter, transmitting signal $x[t]$ and m receiver antennas collaborating with each other to decode the message. Channel coefficients among different receive antennas are independent of each other. The communication channel for each receive antenna is modeled as following.

$$y_j[t] = h_j[t]x[t] + w_j[t], \qquad \text{for } t = 1, \cdots, T, \quad j = 1, \cdots m$$

$h_j[t]$'s are independent of each other and each is normal distributed with mean zero and unit variance. In vector form, we would have:

$$\underline{y}[t] = \underline{h}[t]x[t] + \underline{w}[t]$$

$$\begin{array}{cc} & 1 \\ m & \boxed{\underline{y}[t]} \end{array} = \boxed{\underline{h}[t]}\; x[t] + \boxed{\underline{w}[t]}$$

1.4 Multiple-input and Single-output (MISO) Communications.

In MISO communications there are n transmitting antennas and one receive antennas which receives a linear combination of the transmitted signals from all TX antennas. The channel is modeled as following:

$$\begin{aligned} y[t] &= \sum_{i=1}^{n} h_i^*[t]x_i[t] + w[t], \qquad \text{for } t = 1, \cdots, T \\ &= \underline{h}[t]^H \underline{x}[t] + w[t] \end{aligned}$$

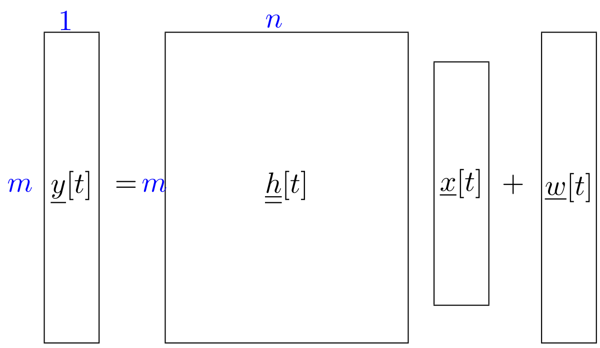
$$y[t] = \begin{array}{cc} & n \\ 1 & \boxed{\underline{h}[t]^H} \end{array} \boxed{\underline{x}[t]} + w[t]$$

Again, $h_i[t]$'s are assumed to be independent, jointly complex gaussian random variables with mean zero and unit variance.

1.5 Multiple-input and Multiple-output (MIMO) Communications.

In MIMO communications there are n transmitting antennas and m receive antennas each receiving a linear combination of the transmitted signals from all TX antennas. The channel is modeled as following:

$$\begin{aligned} y_j[t] &= \sum_{i=1}^{n} h^*_{i,j}[t]x_i[t] + w[t], \qquad\qquad \text{for } t = 1, \cdots, T \\ \underline{y}[t] &= \underline{\underline{h}}[t]\underline{x}[t] + \underline{w}[t] \end{aligned}$$



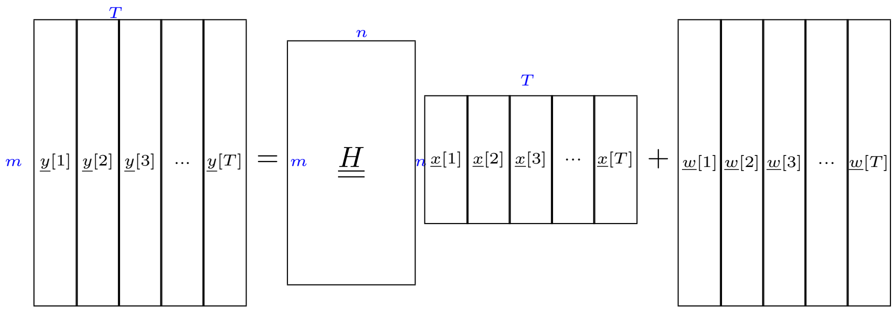
In *flat fading* MIMO communications, the matrix $\underline{\underline{h}}[t]$ is constant over a block of communication i.e., $\underline{\underline{h}}[t] = \underline{\underline{H}}$ for $t = 1, 2, \cdots, T$.

$$\underline{y}[t] = \underline{\underline{H}}\ \underline{x}[t] + \underline{w}[t] \qquad\qquad \text{for } t = 1, \cdots, T$$

$$\underline{\underline{Y}}_{m\times T} = \underline{\underline{H}}_{m\times n} \cdot \underline{\underline{X}}_{n\times T} + \underline{\underline{W}}_{m\times T}$$

1.5.1 Degrees of Freedom (DOF=d)

The received signal in a block of length T in all antennas, $\underline{\underline{Y}} \in \mathcal{C}^{m\times T}$ lives in a space of complex dimension of mT. But not all these dimensions are affected by the input $\underline{\underline{X}}$. The number of dimensions in received signal that are only affected by the input $\underline{\underline{X}}$ (i.e.,



the dimension of subspace that $\underline{\underline{Y}}$ lives in, fixing all parameters and changing only $\underline{\underline{X}}$) is called *the degrees of freedom* of the system.

The importance of this parameter is in the fact that in high SNR regime, degrees of freedom of the system is the *pre-log factor* of capacity of the system i.e., $C(SNR) \approx d \log SNR$

Intuitively, in high SNR regime, the constraint over the input power is not the limiting factor. The limiting factor in communications is the number of dimensions of the received signal that we have control over by changing and designing the input signal.

Chapter 2

Non-coherent Flat Fading MIMO Communications in high SNR

2.1 The Problem Description

Non-coeherent block fading communications was introduced in previous chapter. The question that arises is that what is the degrees of freedom of the system for this setup and how it can be achieved. Meaning that for specific statistics over the matrix $\underline{\underline{H}}$, what is the maximum complex dimension of the matrix $\underline{\underline{Y}}$ that is affected only by changing input signal $\underline{\underline{X}}$ and how should we design the probability distribution over the input signal to achieve this degrees of freedom.

The statistics over the matrix $\underline{\underline{H}} \in \mathcal{C}m \times n$ is assumed IID elements, each gaussian distributed with mean zero and unit variance.

2.2 Optimal Input Distribution

Definition: A random matrix $\mathbf{R} \in \mathcal{C}^{M \times T}$ for $T \geq M$ is called *isotropically distributed (i.d.)* if its distribution is invariant over rotation. i.e., for any deterministic $T \times T$ unitary matrix Q,

$$p(\mathbf{R}) = p(\mathbf{R}Q)$$

.

Lemma: The input distribution that achieves capacity can be written as $\mathbf{X} = \mathbf{A\Theta}$ where $\mathbf{\Theta}$ us an $M \times T$ isotropically distributed unitary matrix, i.e., $\mathbf{\Theta\Theta^H} = I$. $\mathbf{A}$ is an $M \times M$ real diagonal matrix which is independent of $\mathbf{\Theta}$.

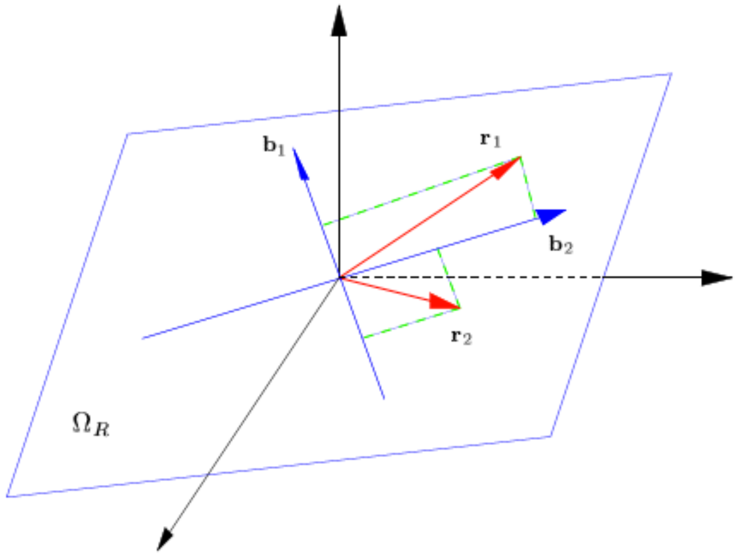
The ith diagonal element of $\mathbf{A}$ represents the power of the signal transmitted in ith antenna. In non-coherent high SNR capacity, it can be proved that constant equal power input is asymptotically optimal $P(\mathbf{A} = \sqrt{T} I_M) = 1$. Intuitively, since there is perfect symmetry between statistics of each antenna, there is no reason to allocate more power to any of them.

2.3 Change of Coordinates

The following transform corresponds to a change of coordinates for a matrix $R \in \mathcal{C}^{M\times T}$ $(T \geq M)$:

$$\begin{aligned} \mathcal{C}^{M\times T} &\rightarrow \mathcal{C}^{M\times M} \times G(T, M) \\ R &\rightarrow (C_R, \Omega_R) \end{aligned}$$

In this transform, Ω_R represents the subspace that is spanned by rows of R which is in Grassman manifold $G(T, M)$ *Grassman manifold* $G(T, M)$ is the set of all M dimensional subspaces of $\mathcal{C}^T$. C_R represents the M rows vectors of matrix R with respect to the canonical basis of Ω_R.



In high SNR regime, the effect of noise in the received signal is very small and we can study the signal $\underline{\underline{Y}}_0 = \underline{\underline{H}}\ \underline{\underline{X}}$ instead of $\underline{\underline{Y}}$.

The subspace spanned by rows of $\underline{\underline{H}}\ \underline{\underline{X}}$ is the same as the subspace spanned by rows of $\underline{\underline{X}}$.

$$\Omega_{HX} = \Omega_X \qquad \text{with probability 1}$$

We could say that the fading coefficients $\underline{\underline{H}}$ change $\underline{\underline{X}}$ only by changing C_X and leave Ω_X unchanged.

Actually, Ω_X is affected only by noise whose power is very small in high SNR regime. But C_X is affected by both noise and channel fading.

We could say

$$\begin{aligned} I(X;Y) &= I(\Omega_X;Y) + I(C_X;Y|\Omega_X) \\ &= I(\Omega_X;Y) \end{aligned}$$

2.4 Channel Capacity

Theorem: If $R \in \mathcal{C}^{M\times T}$ is isotropically distributed (i.e., $\forall Q, p(R) = p(RQ)$), then

$$h(R) = h(C_R) + \log|G(T,M)| + (T-M)\mathbb{E}[\log\det RR^{'}]$$

- $h(C_R) + \log|G(T,M)|$ is differential entropy of R in $\mathcal{C}^{M\times M} \times G(T,M)$.
- $(T-M)\mathbb{E}[\log\det RR^{'}]$ is the Jacobian term for the coordinate change.

Using the above theorem, mutual information $I(X;Y)$ could be approximated in new corrdinate as following:

$$I(X;Y) = h(Y) - h(Y|X)$$

$$\begin{aligned} h(Y|X) &= m\mathbb{E}[\log\det A^2] + m^2\log(\pi e) + m(T-m)\log(\pi e\sigma^2) \\ h(Y) &\approx h(HX) \\ &= h(C_{HX}) + \log|G(T,M)| + (T-m)\mathbb{E}[\log\det HA^2H'] \\ &= h(C_{HX}) + \log|G(T,M)| + (T-m)\mathbb{E}[\log\det HH' + \log\det A^2] \end{aligned}$$

To maximize the mutual information, it is clear that the optimal power allocation is constant for all antennas and the information is conveyed through Ω_X.

Thus, $\mathbf{X} = \mathbf{A\Theta}$ where $P(\mathbf{A} = \sqrt{T}I_M) = 1$ and $\mathbf{\Theta}$ is a unitary matrix with uniform distribution over random unitary matrices in $\mathcal{C}^{T\times T}$.

Chapter 3

Non-coherent Non-Flat Fading MIMO Communications in high SNR

3.1 Problem Setup and Future work

The immediate generalization of the problem introduced in previous chapter concerns non-flat fading channels. The matrix $\underline{\underline{h}}[t]$ in this model is not constant and changes with time. The parameter of interest is again the degrees of freedom of the system and the the optimal input distribution in high SNR.

$$y_j[t] = \sum_{i=1}^{n} h_{ij}[t]x_i[t] + w_j[t] \qquad \text{for } t = 1, ..., T \qquad j = 1, ..., m$$
$$i = 1, ..., n$$

The variation of fading coefficients for each pair of antennas in time $h_{ij}[t]$ for $t = 1, 2, \cdots, T$ is not modeled arbitrary and independently. We assumer some correlation or similarity between consecutive fading coefficients. Looking at the vector of fading coefficient between a pair in time, we assume that for all i, j:

$$\mathbb{E}\left(\begin{bmatrix} h_{ij}(1) \\ h_{ij}(2) \\ \vdots \\ h_{ij}(T) \end{bmatrix} [h_{ij}(1), h_{ij}(2), \cdots, h_{ij}(T)] \right) = K_H$$

We would assume that the $rank(K_H) = Q$ is much smaller that $Q << T$, meaning that there are only a very few independent parameters that determine the sequence of $h_{i,j}[t]$'s for $t = 1, 2, \cdots, T$. It is worth mentioning that when $Q = 1$, there is flat fading and the problem is specialized to the one given in previous chapter.

Determining the degrees of freedom of the system for given m, n, T and $Q << T$ and designing the optimal input distribution is the future work in this area.

Bibliography

[1] L. Zheng and D. Tse *Communication on the Grassman manifold: A geometric approach to the non-coherent multiple-antenna channel.* IEEE Trans. Inf. Th., vol. 48, no.2, pp. 359-383, Feb. 2002.

[2] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.

[3] E. Riegler, V. I. Morgenshtern, G. Durisi, S. Lin, B. Sturmfels, and H. Bolcskei, *Noncoherent SIMO pre-log via resolution of singularities,* in Proc. IEEE Int. Symp. Inf. Th. (ISIT 2011), St. Petersburg, Russia, Aug. 2011, pp. 20202024.

[4] V. I. Morgenshtern, G. Durisi, and H. Bolcskei, *The SIMO pre-log can be larger than the SISO pre-log*, in Proc. IEEE Int. Symp. Inf. Th. (ISIT 2010), Austin, TX, June 2010, pp. 320324.

[5] T. L. Marzetta and B. M. Hochwald, *Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading*, IEEE Trans. Inf. Theory, vol. 45, no. 1, pp. 139157, Jan. 1999.

[6] B. M. Hochwald and T. L. Marzetta, *Unitary spacetime modulation for multiple-antenna communications in Rayleigh flat fading,* IEEE Trans. Inf. Theory, vol. 46, no. 2, pp. 543564, Mar. 2000.