18.338 Final Report: Applications of RMT to Linear Control Systems

Charlotte Kiang

May 16, 2012

1 Introduction

Mathematical control theory is the area of mathematics and deals with the principles underlying the analysis and design of control systems. Control systems themselves are broadly defined as systems whose functions are to influence an object's behavior in order to achieve a desired goal.

The two main branches of control theory are optimization and uncertainty. The problem with which this paper is mainly concerned is one of optimization—that is, formulating an equation or algorithm that accomplishes our goal most efficiently. The system we seek to optimize in this paper is a networked control system that is experiencing delays due to network traffic. Our claim is that by exploring patterns that arise in these random time delays, we can find an equivalence with the random Fibonacci recurrence, and that we can exploit this equivalence to draw further conclusions about control systems.

This problem has been previously explored by Kalmár Nagy, [4] who first posited that a single jump linear control system and a random walk on a self-similar graph whose vertices are the visible points of a plane are equivalent. This paper will walk through this finding and elaborate a bit on the results.

2 Natural occurrence of jump linear patterns in linear control systems

We claim that linear control systems with random time delay follow a jump linear pattern; therefore, we must show why this pattern arises naturally in these systems.

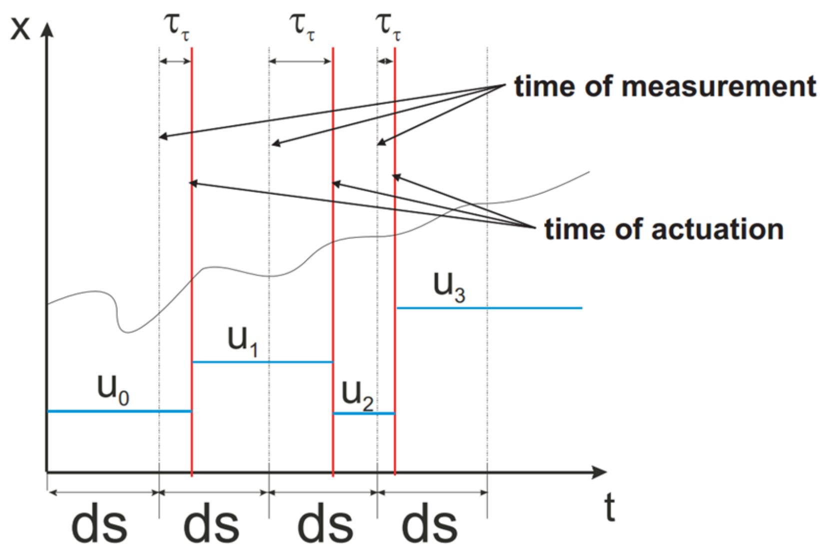
To do this, we consider a basic model of a linear control system with random actuator delays due to communication over a network. Our control system follows the standard state space equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx,$$

with our state evolving continuously. We are assuming here that our control signal is proportional to our output, and our sampling time ds (as seen on graph below) specifies the time instants s_i = ids.

Our terms of interest here are x(t) (our overall state at a given time), and u (our control input).

GRAPH OF A CONTROL SYSTEM WITH RANDOM ACTUATION TIME DELAYS



Graph of random actuation time delays. [4]

We want our actuation times to be the same as our sampling times, but this occurs only in an ideal model. Once we factor in network delays, we must re-express our time as:

$$t_i = s_i + \tau_i \qquad (t_0 = 0, \tau_0 = 0).$$

Note: Here, we are assuming that our delay in actuation is less than our sampling time interval, i.e., $\tau_i < ds$. We acknowledge that this model is highly simplified, but that its observed patterns are still useful.

Using these assumptions, we can derive a mapping between values of the state and control input for consecutive action times. [4] Some manipulation of numbers, as demonstrated by Kalmár Nagy, allows us to obtain the following mapping:

$$\begin{pmatrix} x \\ u \end{pmatrix}_{i+1} = \underbrace{\begin{pmatrix} C(\tau_{i+1}, \tau_i) & D(\tau_{i+1}, \tau_i) \\ E(\tau_i) & F(\tau_i) \end{pmatrix}} \begin{pmatrix} x \\ u \end{pmatrix}_i$$

Thus, we now have a recursive relationship between our state and control input at any given time. We can then rewrite this as:

$$\begin{pmatrix} x \\ u \end{pmatrix}_n = \prod_{i=0}^{n} \mathcal{A}_i \begin{pmatrix} x \\ u \end{pmatrix}_0 .$$

We can see from this that each of our terms is dependent upon the previous one in the sequence. To account for this dependency, we can describe the distribution of these values over an associated Markov chain.

By definition, this gives us a linear jump system.

3 Consequences of linear jump patterns

The existence of linear jump patterns within linear control systems allows us to run Monte Carlo simulations to obtain information about the stability of a system. (However, the goal of this paper is instead to understand the underlying principles that govern this class of problem—so our focus is on drawing parallels to this finding.)

4 Fibonacci series and graph

If we take the well-known Fibonacci recurrence,

$x_{n+1} = x_n \pm x_{n-1}$,

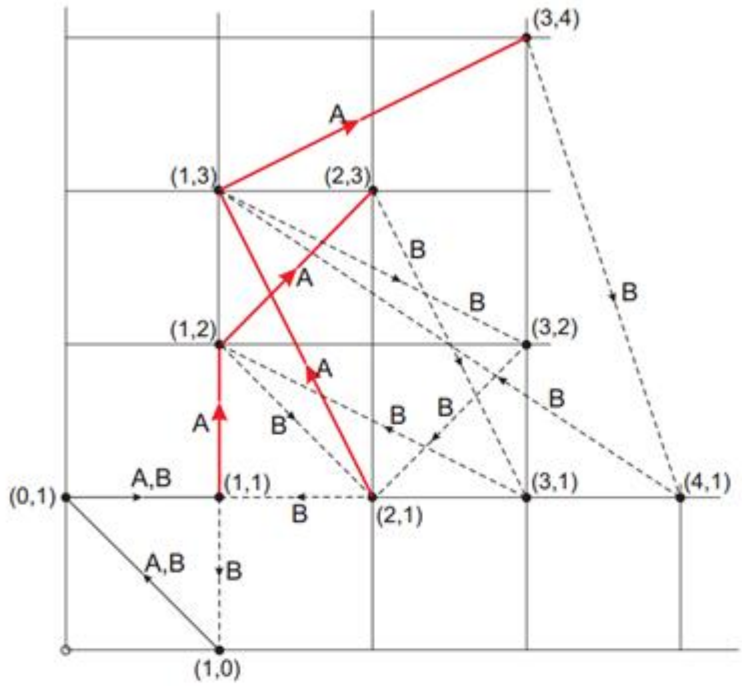
we can rewrite it using a two-dimensional map:

$$\begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} = D_n \begin{pmatrix} x_{n-2} \\ x_{n-1} \end{pmatrix},$$

where the coefficient matrix D_n is chosen randomly as either A or B with probability ½, and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Recall that our original goal was to elucidate the symmetry between our linear control system equation and this Fibonacci one. In order to do this, we must examine the geometry induced by the Fibonacci series.



Graph of the random Fibonacci maps on the first quadrant. [4]

Let us consider the following mappings of lattice points to lattice points (i.e., points with integer coordinates).

$A: (i, j) \rightarrow (j, i + j)$
$B: (i, j) \rightarrow (j, i - j)$

We can now modify this recurrence so that it acts only on the first quadrant of Z^2.

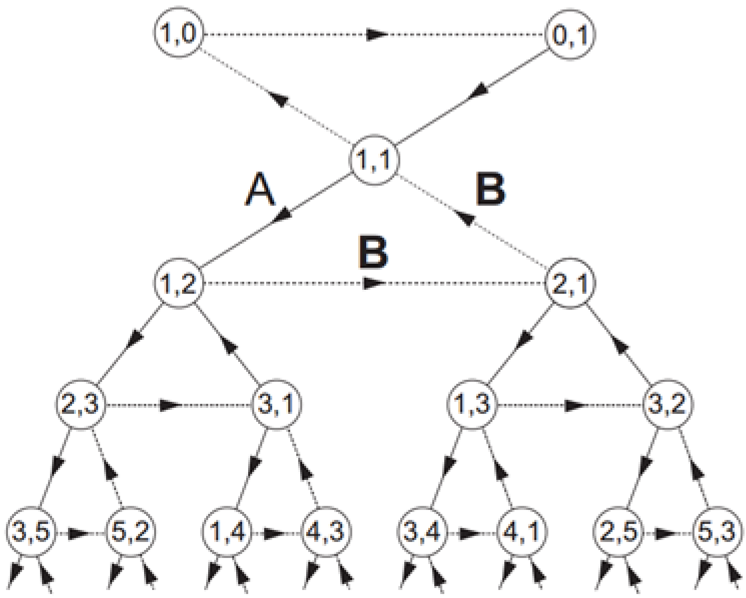
Our modified mappings will then be:

$A: (i, j) \rightarrow (j, i + j)$
$B: (i, j) \rightarrow (j, |i - j|) \qquad i,j > 0.$

If we repeat this mapping starting from (1, 1), we obtain the graph above. (A consequence of this graph is that all vertices are relatively prime, although I have yet to prove this or find a proof for it.) [4] This graph is a directed graph otherwise known as the Fibonacci graph.

5 Basic topology of the Fibonacci graph

The graph shown above can be "unfolded" to produce a new graph with nodes of the same values.



"Tree" version of the Fibonacci graph. [4]

We will define a point (i, j) as visible if gcd(i, j) = 1. We will also say that a point (k, l) is reachable from another point (i, j) if there exists a series of transformations (A and B as defined in section 4) that takes (i, j) to (k, l). This is equivalent to requiring that all points be relatively prime. In section 4, we posited that this was true of every node in the Fibonacci graph, so all points on our graph are visible.

From these statements, we can conclude that each point on our graph has an "address"—i.e., a series of transformations that take (i, j) to (1, 1).

This is equivalent to our earlier statement about linear jump systems:

$$\begin{pmatrix} x \\ u \end{pmatrix}_n = \prod_{i=0}^{n} \mathcal{A}_i \begin{pmatrix} x \\ u \end{pmatrix}_0 .$$

6 Conclusion

Now that we have established an equivalence between linear control systems and the Fibonacci graph, we can use the graph as a model for many other problems in control theory. [4] This strongly suggests that graph theory, combinatorics, statistical physics and even number theory have applications in control theory.

The model obtained in section 2 also provides motivation for further analysis of the transformation matrix's eigenvalues. We saw that the stability of our system (i.e., rate of convergence/divergence) would be determined by the growth/decay of the associate infinite random matrix product. This suggests that we can use eigenfunctions as our basis for diagonalization and separation of variables. It is conceivable that we can decouple our problem in a manner similar to Fourier's problem of heat conduction in a solid bar with zero temperature at both ends. [2] We can also use our eigenvalues from our model in section 2 to glean information about our system's resonance (i.e., level of response to selected inputs).

References

[1] Czorniak, Adam and Swierniak, Adrzej. "On controllability of discrete time linear jump systems." http://mbi.osu.edu/publications/reports/techreport_08.pdf. Accessed April 15, 2012.

[2] Embree, Mark and Trefethen, Lloyd. Spectra and Pseudospectra. Princeton University Press, 2005.

[3] Gupta, Vijay and Murray, Richard M. "Lecture Summary: Markov Linear Jump Systems." (http://www.eeci-institute.edu/pdf/M010/Lecture_mjls.pdf) European Embedded Control Insitute. Accessed May 5, 2012.

[4] Kalmár Nagy, Tamás. "Random Matrix Theory Approach to Linear Control Systems With Network-Induced Delays." *Proceedings of the 46th IEEE Conference on Decision and Control*, Dec 12-14 2007: 5210-5214. Accessed online.

[5] Sontag, Eduardo. Mathematical Control Theory: Deterministic Finite Dimensional Systems. Second Edition, Springer, New York, 1998.