Spectral Perturbation of Small-World Networks with Application to Brain Disease Detection

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Outline

- Overview
- Spectra of Small-World Networks
- Perturbation of Small-World Network
- Conclusion

Overview

- Small-World Network

- Capture both order and randomness
- Real systems: social networks, power grids, neural networks, …

- Spectrum of SW Network

- Eigenvalue distribution
- Perturbation model
- Brain disease detection: Alzheimer’s disease

Spectra of Small-World Networks

- Key featurers
  - most nodes are not neighbors of one another
  - average distance is small
- Construction by rewiring a regular graph

Figure 1: Diagram showing the relation among small-world network, regular network, and random network[1].

Network Construction

- Two steps
  - Regular ring graph: N nodes each degree k
  - Rewiring each edge with prob. P
- Measurable quantities

- L_{sw} : average shortest path length
- C_{sw}: fraction of pairs of neighbors of a node which are also neighbors to each other

Small-World Metrics

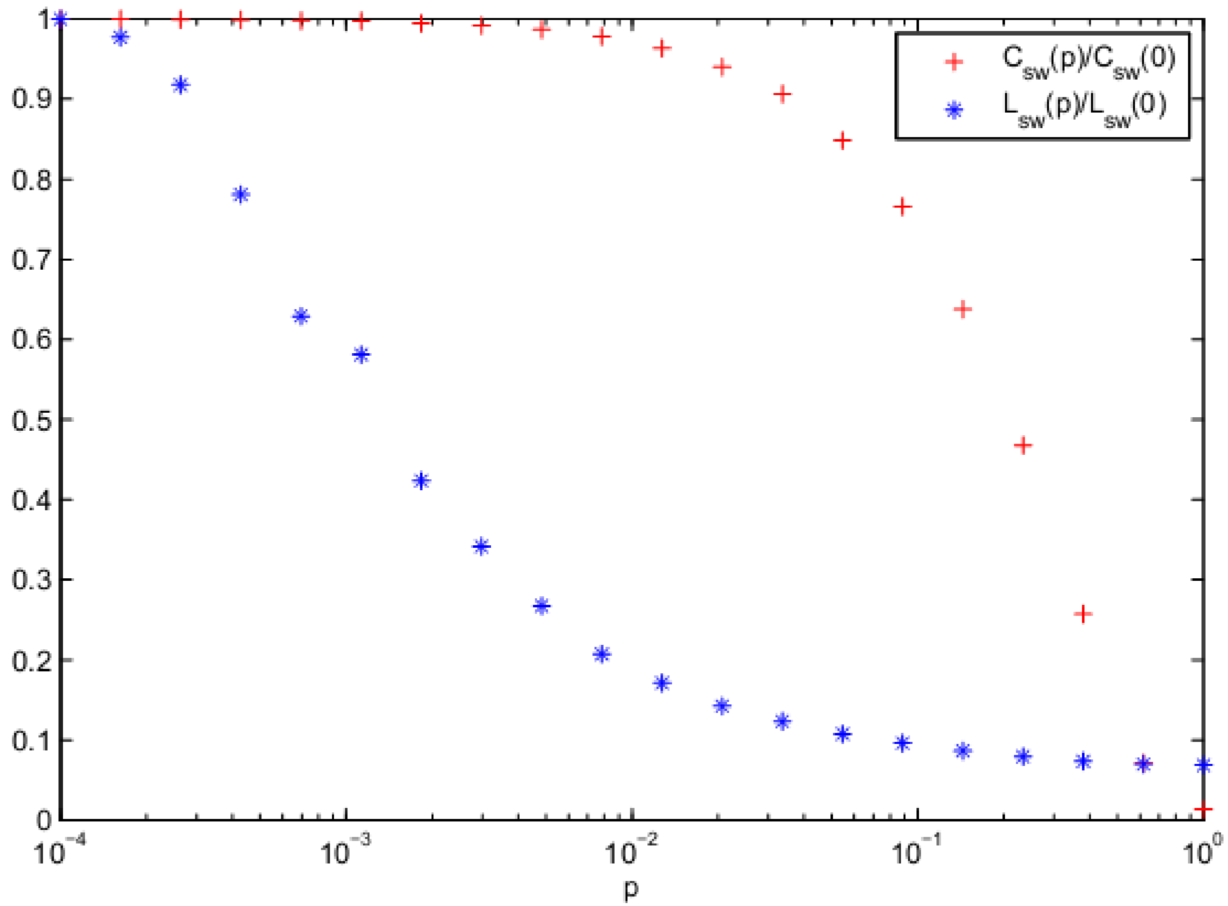

Figure 2: The characteristic path length L_{sw} and clustering coefficient C_{sw} against p ranging from 10^{-4} to 1. The data are averages over 20 random realizations of rewiring and are normalized by dividing their values at $p = 0$, respectively.

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Spectra Properties

- Adjacency & Degree matrix

adjacency matrix $A = (a_{ij})$

degree matrix D

- Laplacian matrix

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2}$$

Spectral Properties – Cont.

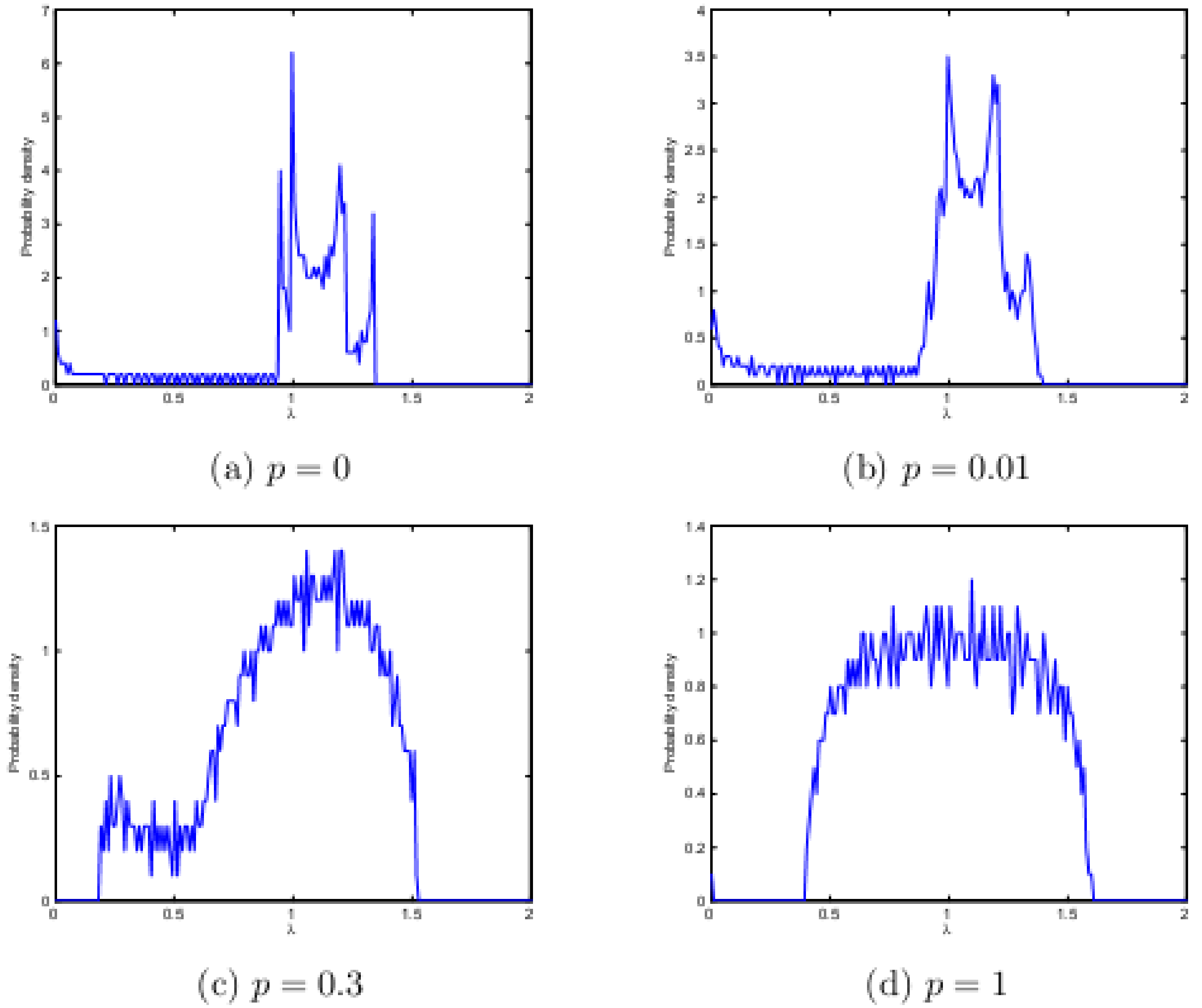


Figure 3: Eigenvalue distribution of small-world network with different level of rewiring probabilities p. The number of nodes $N = 1000$ and the degree of each node $k = 10$ are fixed for all four cases.

Spectra Properties

- Structure of eigenvectors

  - inverse participation ratio of normalized eigenvector $v_i = \left(v_i^{(1)}, v_i^{(2)}, \cdots, v_i^{(N)}\right)$

$$I_i = \sum_{j=1}^{N} \left[v_i^{(j)}\right]^4$$

- Particular: eigenvector with identical entries/ a single non-zero component

Spectral Properties – Cont.

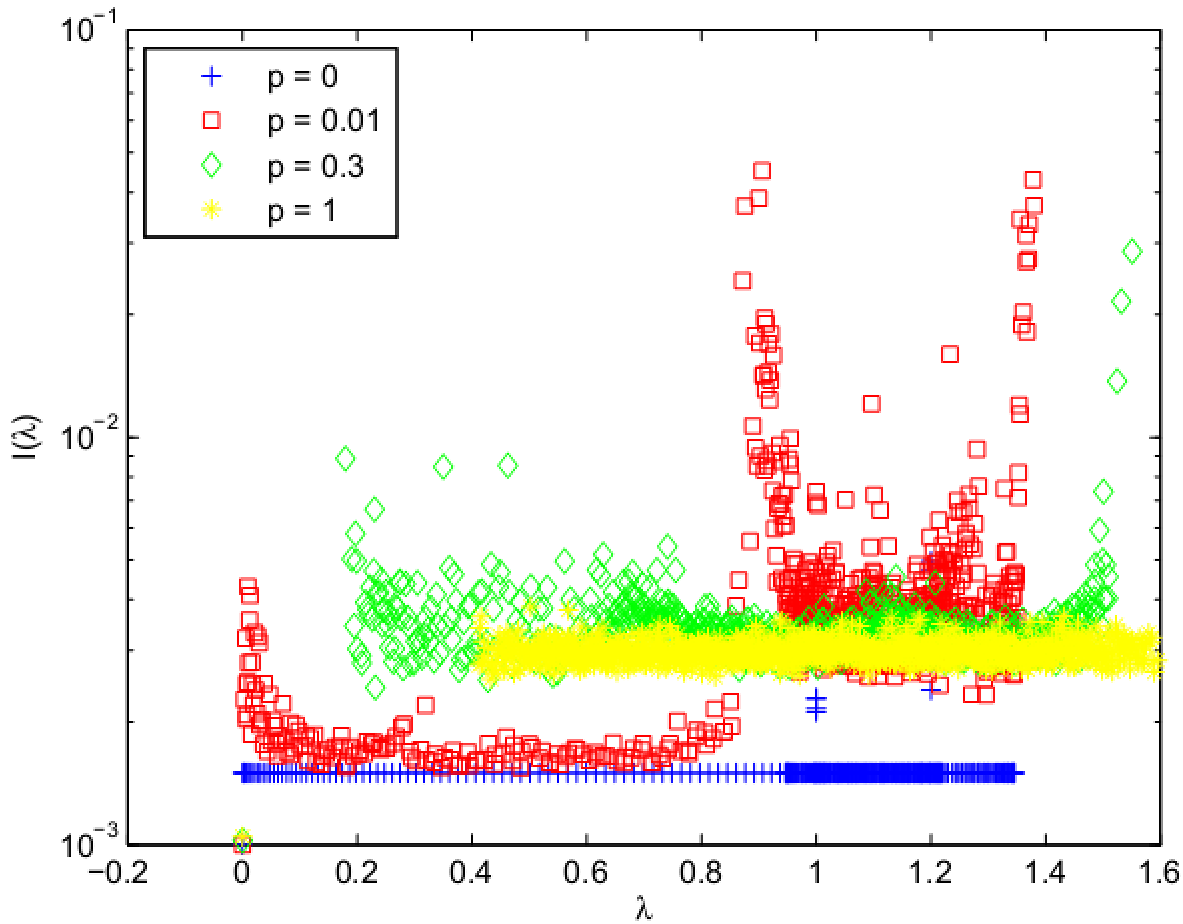


Figure 4: The inverse participation ratios of eigenvectors for small-world networks with different rewiring probability p. $N = 1000, k = 10$ are fixed for all cases.

Perturbation of Small-World Network

- Brain Network Reconstruction
  - Alzheimer's disease (AD), mild cognitive impairment (MCI), normal aging (NA)
  - Brain network alternation
  - Neuroimaging: MRI, PET, fMRI,…
- Sparse inverse covariance estimation (SICE)

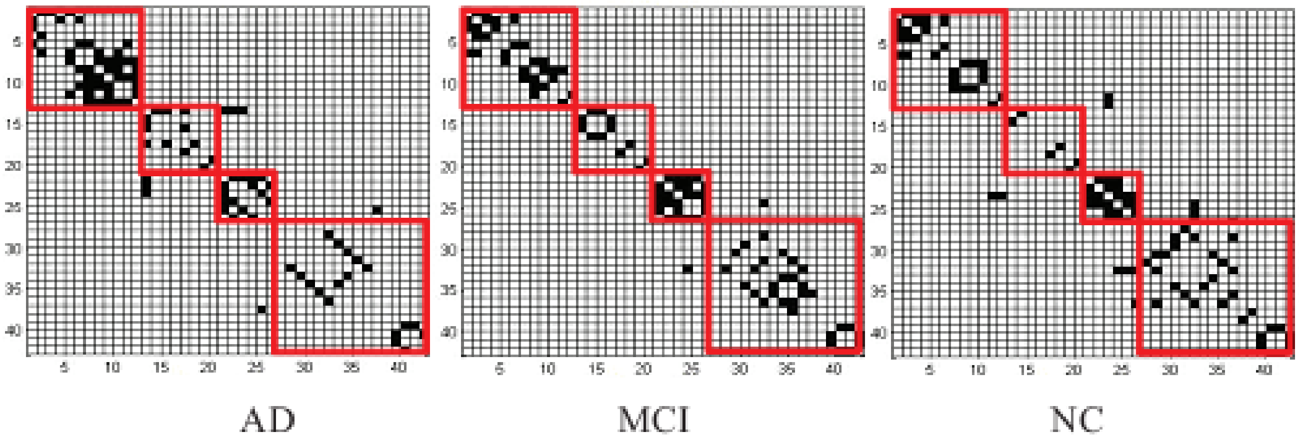
voxel values $\{X_1, X_2, \cdots, X_M\}$ *Multivariant Gaussian*

$$\hat{\Theta} = \text{argmax}_{\Theta>0} \log(\det(\Theta)) - \text{tr}(S\Theta) - \gamma\|\text{vec}(\Theta)\|_1,$$

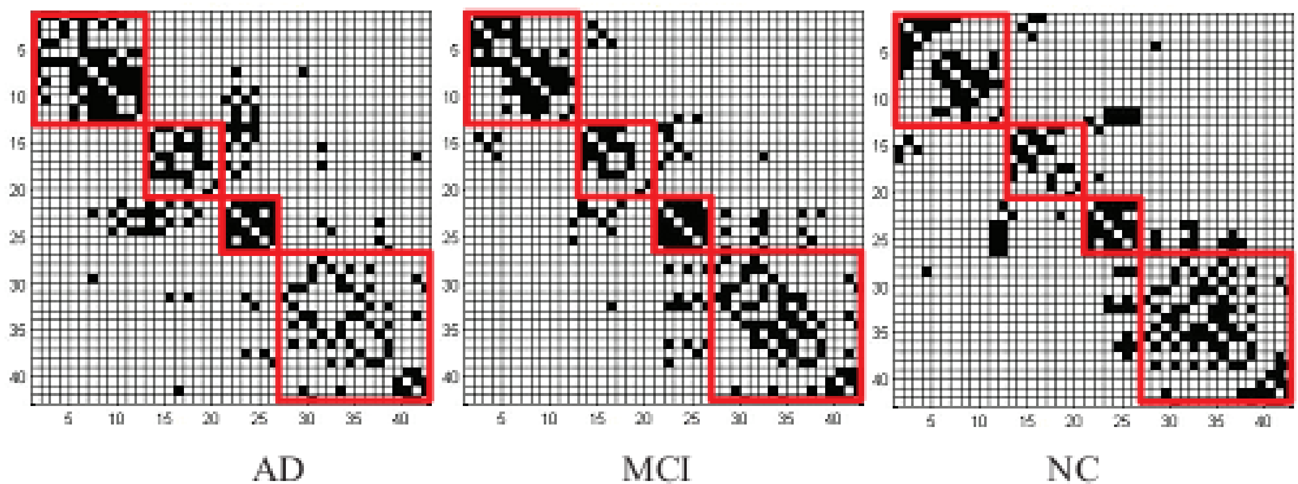
$\Theta = \Sigma^{-1}$: inverse covariance matrix

S is the sample covariance matrix

Averaged Human Brain Networks



(a) SICE-based brain connectivity networks (total number of edges equal to 50)



(b) SICE-based brain connectivity networks (total number of edges equal to 120)

Figure 5: Reconstructed brain networks with two different number of edges. The regularization factors for each category of images are adaptively chosen in order to make their amounts of edges identical [7].

Extension of SW Network

- For node j, assign degree k_j
- A shift s_j for the center of the neighborhood
- Different rewiring prob. P_j

SW(N, k, s, p) with k, s, p be vectors

Network Classify by Spectral Correlations

- Perturbation from a base network

$$\tilde{a}_{ij} = \begin{cases} 1 - a_{ij}, & \text{with prob. } p_t; \\ a_{ij}, & \text{with prob. } 1 - p_t \end{cases}$$

- Spectral correlation (eigenvalues)

$$\rho(u, v) \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{N} u_i v_i}{\sum_{i=1}^{N} u_i^2 \sum_{i=1}^{N} v_i^2}$$

Results on Real Data

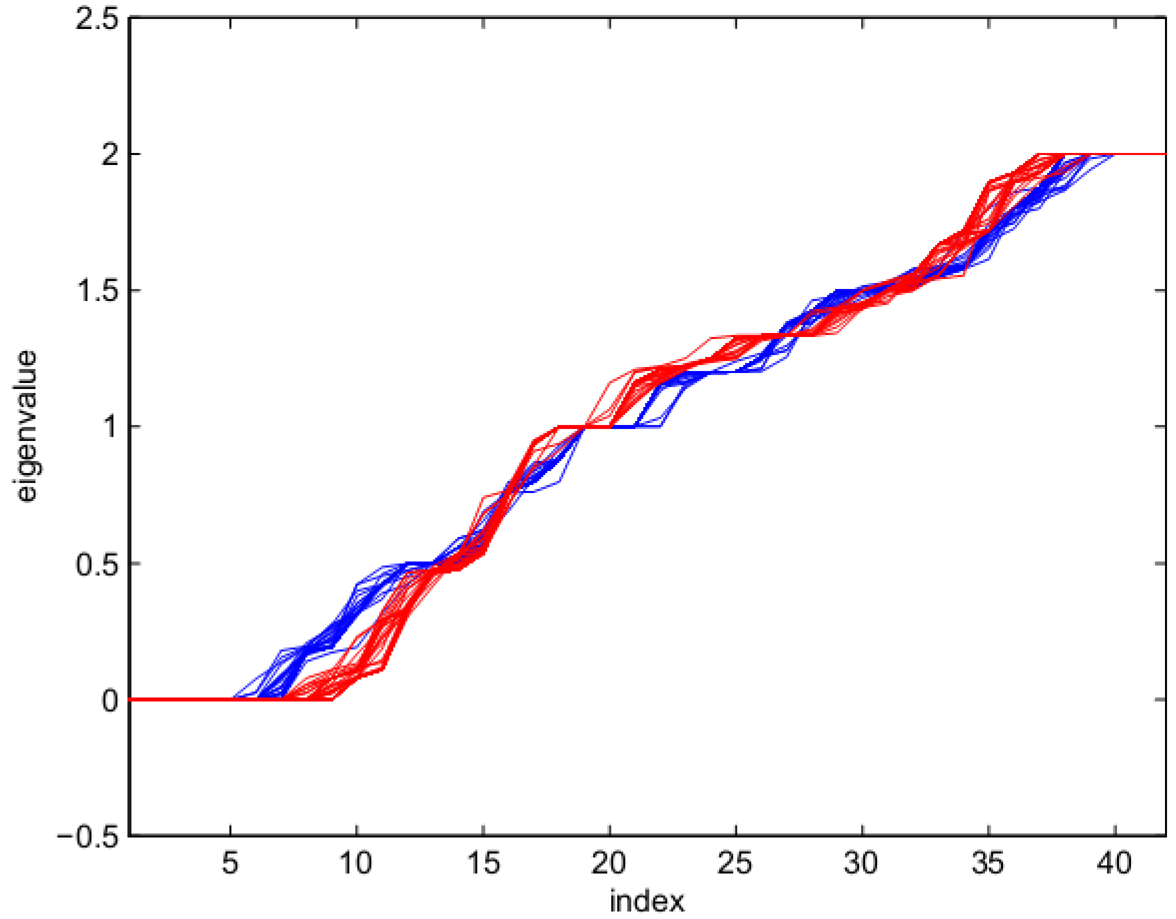
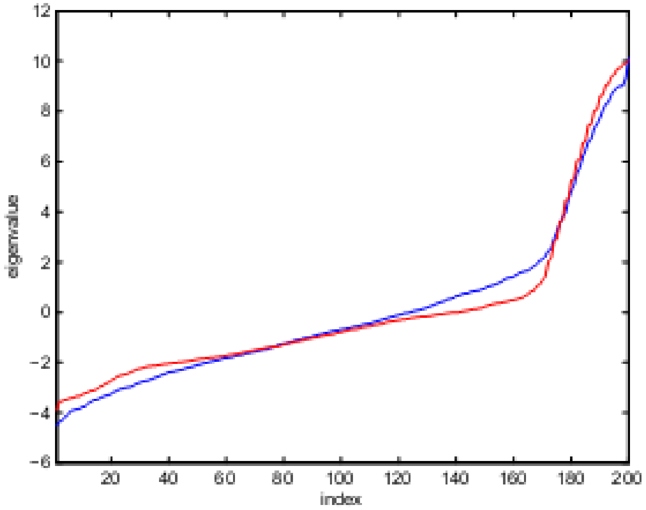
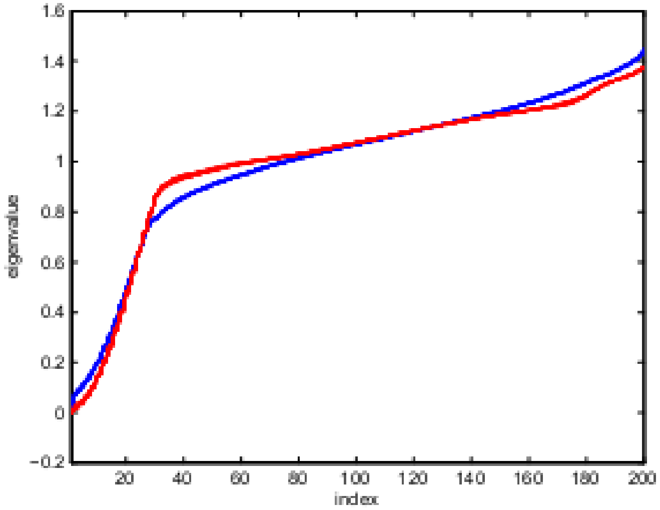


Figure 6: The perturbed eigenvalue distributions for AD (blue lines) and NC (red lines). The perturbation probability p_t for each edge is 10^{-3}.

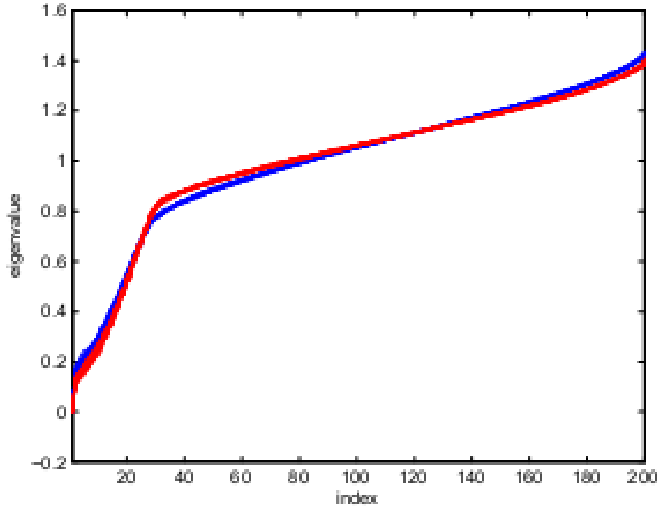
Results on Standard SW Networks



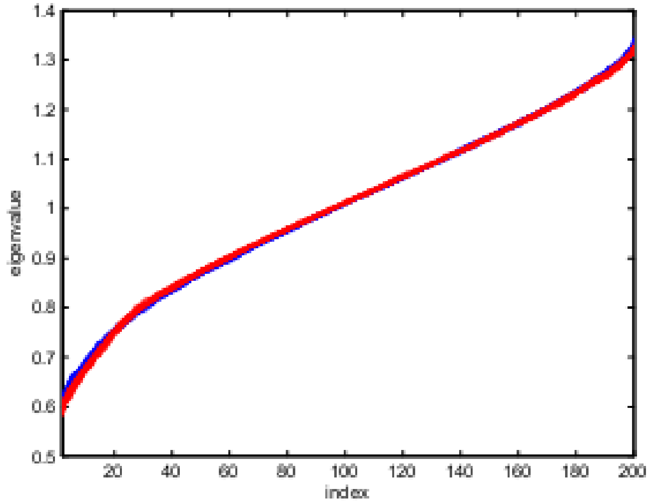
(a) $p_t = 0$



(b) $p_t = 0.001$



(c) $p_t = 0.01$



(d) $p_t = 0.1$

Summary

- Studied spectral properties of small-world networks
- Perturbation of the network
- Classification of real networks