18.338 Eigenvalues of Random Matrices

Problem Set 4

Due Date: Wed Mar. 14, 2012

Reading and Notes

Read chapter *11.1-11.4* of the class notes. Do whatever you like but please remember to provide comments for the course notes. Start thinking about projects. Suggestions will be online soon. Please hand in hardcopy or email (including both the answers to the problems and comments to the readings) to Bernie Wang (ywang02@mit.edu).

Homework

1. (M) Show that the probability that a random $n \times n$ matrix A (`randn(n)`) has all real eigenvalues is
$$\mathbb{P}(A \text{ has all real eigenvalues}) = \frac{1}{2^{n(n-1)/4}}. \tag{1}$$
To derive the results, one can follow
   (a) Work out the Jacobian of the real Schur decomposition $\mathrm{d}A = \mathrm{d}(QRQ^T)$;
   (b) Write out joint density of A and integrate over the set that the eigenvalues of A are real;
   (c) Use the joint density of GOE to obtain the final answer.
2. (C) Using Monte Carlo, estimate the expected number of real eigenvalues of `randn(n)`. Perhaps try $n = 25, 100, 400$ to guess the pattern.
3. (M) Exercise 9.1 (p101)
4. (M) Let V be any $n \times p$ matrix with $V^T V = \mathbb{I}_p$ and F is any $n \times p$ matrix, prove that
$$(F^T \mathrm{d}x)^\wedge = \det(F^T V)(V^T \mathrm{d}x)^\wedge,$$
thereby showing that
$$\int f(x)\mathrm{d}(\text{surface}) \approx \sum f(x^{(i)}) Pl(F^{(i)})^T Pl(V^{(i)}) = \sum f(x^{(i)}) \det((F^{(i)})^T V^{(i)}),$$
make sense.
5. (M) Let $\boldsymbol{q}$ be uniformly distributed on the sphere on $\mathbb{R}^n$. Let $e_1, e_2, \cdots, e_n$ be nonnegative even integers. Find a formula for
$$\mathbb{E}[q_1^{e_1}, \cdots, q_n^{e_n}],$$
(probably using Gamma functions) and that $\boldsymbol{q} = \boldsymbol{x}/\|\boldsymbol{x}\|$ where `x=randn(n,1)`.
6. (C) This is from problem set 2, but I am now sure it is doable. Write a MATLAB code that computes the GOE version of the finite level densities for even n. There is a mathematica code (see supplementary material) based on symbolic integration. The MATLAB code for computing ϕ's is in the notes. You will need the following
$$\begin{aligned}
&H_n'(x) = 2nH_{n-1}(x)\\
&\int_0^x e^{-y^2} H_n(y)\mathrm{d}y = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)\\
&H_{2n}(0) = (-1)^n 2^n (2n-1)!!\\
&H_{2n+1}(0) = 0.
\end{aligned}$$