THEOREM 3.3.1. Let A and B be independent, where A is $W_m(n_1, \Sigma)$ and B is $W_m(n_2, \Sigma)$, with $n_1 > m-1$, $n_2 > m-1$. Put $A+B=T'T$ where T is an upper-triangular $m \times m$ matrix with positive diagonal elements. Let U be the $m \times m$ symmetric matrix defined by $A = T'UT$. Then $A+B$ and U are independent; $A+B$ is $W_m(n_1+n_2, \Sigma)$ and the density function of U is

(1)

$$\frac{\Gamma_m\left[\frac{1}{2}(n_1+n_2)\right]}{\Gamma_m\left(\frac{1}{2}n_1\right)\Gamma_m\left(\frac{1}{2}n_2\right)}(\det U)^{(n_1-m-1)/2}\det(I_m-U)^{(n_2-m-1)/2} \qquad (0<U<I_m),$$

where $0<U<I_m$ means that $U>0$ (i.e., U is positive definite) and $I_m - U > 0$.

Proof. The joint density of A and B is

$$\frac{2^{-m(n_1+n_2)/2}(\det\Sigma)^{-(n_1+n_2)/2}}{\Gamma_m\left(\frac{1}{2}n_1\right)\Gamma_m\left(\frac{1}{2}n_2\right)}\operatorname{etr}\left[-\tfrac{1}{2}\Sigma^{-1}(A+B)\right](\det A)^{(n_1-m-1)/2}$$

$$\cdot(\det B)^{(n_2-m-1)/2}(dA)(dB).$$

First transform to the joint density of $C=A+B$ and A. Noting that $(dA)\wedge(dB)=(dA)\wedge(dC)$ (i.e., the Jacobian is 1), the joint density of C and A is

$$(2) \qquad \frac{2^{-m(n_1+n_2)/2}(\det\Sigma)^{-(n_1+n_2)/2}}{\Gamma_m\left(\frac{1}{2}n_1\right)\Gamma_m\left(\frac{1}{2}n_2\right)}\operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}C\right)(\det A)^{(n_1-m-1)/2}$$

$$\cdot\det(C-A)^{(n_2-m-1)/2}(dA)(dC)$$

Now put $C=T'T$, where T is upper-triangular, and $A=T'UT$. Remembering that T is a function of C alone we have

$$(dA)\wedge(dC)=(T'\,dUT)\wedge(d(T'T))$$

$$=(\det T)^{m+1}(dU)\wedge(d(T'T))$$

$$=\det(T'T)^{(m+1)/2}(dU)\wedge(d(T'T)),$$

where Theorem 2.1.6 has been used. Now substitute for C, A, and $(dA)(dC)$ in (2) using $\det A=\det(T'T)\det U$ and $\det(C-A)=\det(T'T)\det(I-U)$. Then the joint density function of $T'T$ and U is

$$\frac{2^{-m(n_1+n_2)/2}(\det\Sigma)^{-(n_1+n_2)/2}}{\Gamma_m\left[\frac{1}{2}(n_1+n_2)\right]}\operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}T'T\right)\det(T'T)^{(n_1+n_2-m-1)/2}$$

$$\cdot\frac{\Gamma_m\left[\frac{1}{2}(n_1+n_2)\right]}{\Gamma_m\left(\frac{1}{2}n_1\right)\Gamma_m\left(\frac{1}{2}n_2\right)}(\det U)^{(n_1-m-1)/2}\det(I-U)^{(n_2-m-1)/2},$$

which shows that $T'T=C=A+B$ is $W_m(n_1+n_2,\Sigma)$ and is independent of U, where U has the density function (1).

DEFINITION 3.3.2. A matrix U with density function (1) is said to have the multivariate beta distribution with parameters $\frac{1}{2}n_1$ and $\frac{1}{2}n_2$, and we will write that U is $\text{Beta}_m(\frac{1}{2}n_1,\frac{1}{2}n_2)$. It is obvious that if U is $\text{Beta}_m(\frac{1}{2}n_1,\frac{1}{2}n_2)$ then I_m-U is $\text{Beta}_m(\frac{1}{2}n_2,\frac{1}{2}n_1)$.

The multivariate beta distribution generalizes the usual beta distribution in much the same way that the Wishart distribution generalizes the χ^2 distribution. Some of its properties are similar to those of the Wishart distribution. As an example it was shown in Theorem 3.2.14 that if A is $W_m(n,I_m)$ and is written as $A=T'T$, where T is upper-triangular, then $t_{11},t_{22},\dots,t_{mm}$ are all independent and t_{ii}^2 is χ^2_{n-i+1}. A similar type of result holds for the multivariate beta distribution as the following theorem, due to Kshirsagar (1961, 1972), shows.

THEOREM 3.3.3. If U is $\text{Beta}_m(\frac{1}{2}n_1,\frac{1}{2}n_2)$ and $U=T'T$, where T is upper-triangular then $t_{11},\dots,t_{mm}$ are all independent and t_{ii}^2 is $\text{beta}[\frac{1}{2}(n_1-i+1),\frac{1}{2}n_2]$; $i=1,\dots,m$.

Proof. In the density function (1) for U, make the change of variables $U=T'T$; then

$$\det U=\det T'T=\prod_{i=1}^{m}t_{ii}^2$$

and, from Theorem 2.1.9,

$$(dU)=2^m\prod_{i=1}^{m}t_{ii}^{m+1-i}\bigwedge_{i\le j}^{m}dt_{ij}$$