18.338 Eigenvalues of Random Matrices

Problem Set 3

Due Date: Wed Feb. 29, 2012

Reading and Notes

Read chapter *6.1-6.5, 9.3-9.5* of the class notes.

Comments on the readings are a *required* part of the homework! Please comments about what you have read. Comments should include any errors you catch, or stylistic comments. If the material is hard to follow, I want to know. Okay to let me know where the writing became hard to follow.

Please hand in hardcopy or email (including both the answers to the problems and comments to the readings) to Bernie Wang (ywang02@mit.edu).

Homework

Do either of the following two problems. (Computational/Mathematical problems are denoted as C/M. Exercise with numbers and pages are from the class notes.)

1. (C) The purpose of this exercise is to compute densities of quantities related to random matrices numerically and with Monte-Carlo experiments. The two quantities you would be required to compute are

   - the density of the largest principal angle between random subspaces
   - the distribution of the largest eigenvalue of a Wishart matrix

   For both exercises you would need the **mhg** package[1]. Make sure **mhg** and **mhgi** work and compile them if necessary by typing, e.g., `mex mhgi.c` at the command prompt. Test to make sure they work. Type `mhgi(5,2,1,1,2,2)` and `hg(7,2,1.1,1.1,[1,3,3])`. You should get the values 42.8667 and 656.5694, respectively.

   (a) **Largest principal angle** Refer to the following paper[2]. To get the largest principal angle between two p-planes in $\mathbb{R}^n$, start with a random matrix `A=randn(n,p)`. If $A = QR$ is its QR decomposition, then the smallest singular value of the upper $p \times p$ submatrix of Q is the cosine our desired angle, i.e., `theta = acos(min(svd(Q(1:p,1:p))))`. Pick small values of n and p, e.g., $n = 7, p = 3$, and generate about 10,000 random samples of that angle. Histogram the result. Compare it with the density of the angle given analytically by Theorem 1 in the above paper. Plot both the empirical and analytical results on the same graph.

   (b) **Largest eigenvalue of a Wishart matrix** For this exercise we need to generate a large number of Wishart matrices and compare the distribution of the largest eigenvalue with the theoretical prediction. Again, pick small values for n and p, e.g., $n = 3, p = 4$. The value of n must not exceed that of p. To generate an $n \times n$ random Wishart matrix A with p degrees of freedom and covariance matrix S (S is also $n \times n$), one does this: `B=randn(p,n)*sqrt(S); A=B'*B`. Pick S to be symmetric positive definite but otherwise arbitrary. You may assume it is diagonal without any loss of generality. Generate about 10,000 Wishart matirces and get a vector of the largest eigenvalues. Compare this empirical result with the analytical prediction given, e.g., by the last equation on page 842 of the paper in Handout 2 titled *The Efficient Evaluation of the Hypergeometric Function of a Matrix Argument*[3]

[1] downloadable at http://www-math.mit.edu/~plamen/software/mhgref.html
[2] Downloadable at http://math.mit.edu/~plamen/files/AEK.pdf
[3] Also downloadable at http://math.mit.edu/~plamen/files/hyper.pdf.

2. (M) Derive the joint element density for $Y_1^T Y_1$ of the Jacobi ensemble. Begin with the joint density of two Wishart matrices $A^T A$ and $B^T B$. Convert this to a joint density of $A^T A$ and $A^T A + B^T B$. Using the Kronecker product Jacobian compute the joint density of $Y_1^T Y_1$ and $R^T R$, then integrate out $R^T R$. OK to look at Page 109 and 110 of Muirhead, which will be posted online.