18.338 Eigenvalues of Random Matrices

Problem Set 2

Due Date: Wed Feb. 22, 2012

Reading and Notes

Read chapter *2.1-2.5, 4.2.1-4.2.4, 9.1-9.2* of the class notes.

Comments on the readings are a *required* part of the homework! Please comments about what you have read. Comments should include any errors you catch, or stylistic comments. If the material is hard to follow, I want to know. Okay to let me know where the writing became hard to follow.

Please hand in hardcopy or email (including both the answers to the problems and comments to the readings) to Bernie Wang (ywang02@mit.edu).

Homework

Do at least four out of the following problems (Computational/Mathematical problems are denoted as C/M. Exercise with numbers and pages are from the class notes. Notice that there are two "super" computational problems marked as C* which count for four problems each.)

1. (MC) Exercise 2.3 (p23): [M] Think of this as a combinatorial problem. [C, not really] Think of this as a Dynamic programming problem.
2. (M) Exercise 2.6 (p24)
3. (C) Exercise 3.3 (p32)
4. (M) Exercise 4.1 (p41)
5. (C) Experimentally verify the Marcenko-Pastur law (see Sec. 4.2 in the notes) when $r = 1/2$.
6. (M) The polar decomposition is $A = QS$ where $A \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$ is orthogonal, and S is positive semi-definite symmetric. Find the Jacobian (perhaps using wedge notation) for this decomposition.
7. (M) When n is even, a real antisymmetric matrix A can be factored $Q^T DQ$ where D has diagonal blocks
   $$\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}.$$
   This decomposition is not unique, but can be made essentially unique by making `Q(1,2:2:n)` zero (the even columns of the first row of Q). Find the Jacobian of this change of variables.
8. (MC) Consider the Wishart matrix `A=randn(m,n)`, $m \geqslant n$, and `W=A'A`.

   Mathematically or computationally through Monte Carlo: Show that an off-diagonal element of a Wishart matrix is the product of a Gaussian and a χ_m distribution for any m, n. For MATLAB users, try out `qqplot` (which I like, though the tails drift) and `kstest` (which never works very well, I think.) Similarly show that the diagonals are χ_m^2.

   Show that the diagonal elements are independent through math or monte carlo. Also show that the off-diagonal elements that share a row or column index are not. One can check for independence by seeing that the moments multiply or arguing logically.

9. (MC) This exercise is inspired by the work of two students in our class this week:

   We can generate vectors q that are uniform on the sphere by computing `x=randn(n,1);q=x/norm(x)`. Show that for finite n, `sqrt(n)q(1)` is not normally distributed but has mean 0 and variance 1. Show that as $n \to \infty$, $\sqrt{n}q(1)$ approaches the normal distribution by computing the moments mathematically or using `qqplot` in MATLAB. (Mathematically, we are taking the moments of a beta distribution)

   For finite n, $\sqrt{n}q(1)$ and $\sqrt{n}q(2)$ are not independent. Show as $n \to \infty$ asymptotically they are. Mathematically, one can integrate over spheres. OK to ask me for a hint. Computationally one can plot the density, or monte carlo enough moments to be convincing.

10. (C*) This computational problem counts as four, so it is a complete and valid homework on its own.

    Chapter *5.3.1* is not yet on the reading but please take a look at Chapter *5.3.1*. We are going to ask you to write a complete version of a finite Wishart law by analogy of the finite GUE law. For the monte carlo experiment, one needs to take the squares of the `svd` of `randn(m,n)` for $m \geqslant n$.

    For the orthogonal polynomials one needs to obtain a symmetric tridiagonal matrix that encodes a three term recurrence for orthonormal Laguerre polynomials. The Laguerre parameter should be $m - n$.

    Brian Sutton's thesis [1] in Formulas (3.3.8) and (3.3.9) on Page 37 may be handy in constructing the tridiagonal.

    This is a little researchy, since I have not seen anyone do this yet, but it would great to have a nicely coded version of all this. I'd be happy to help.

11. (C*) This is also a resarchy problem worth four computational problems. Write a MATLAB code that computes the GOE version of the finite level densities. There is a mathematica code (see the attached notes) based on symbolic integration. I believe that a numerical version should be doable, but I havent given this enough thought at this point. Happy to talk about this as well.

[1] Downloadable at http://faculty.rmc.edu/bsutton/publications/sutton-thesis.pdf