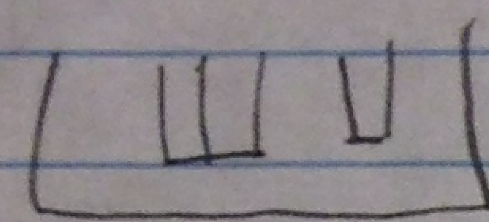
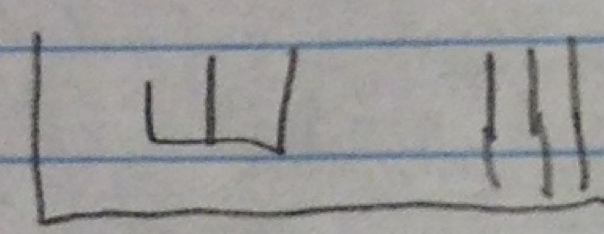
Non crossing Partitions $\qquad NC(n)$

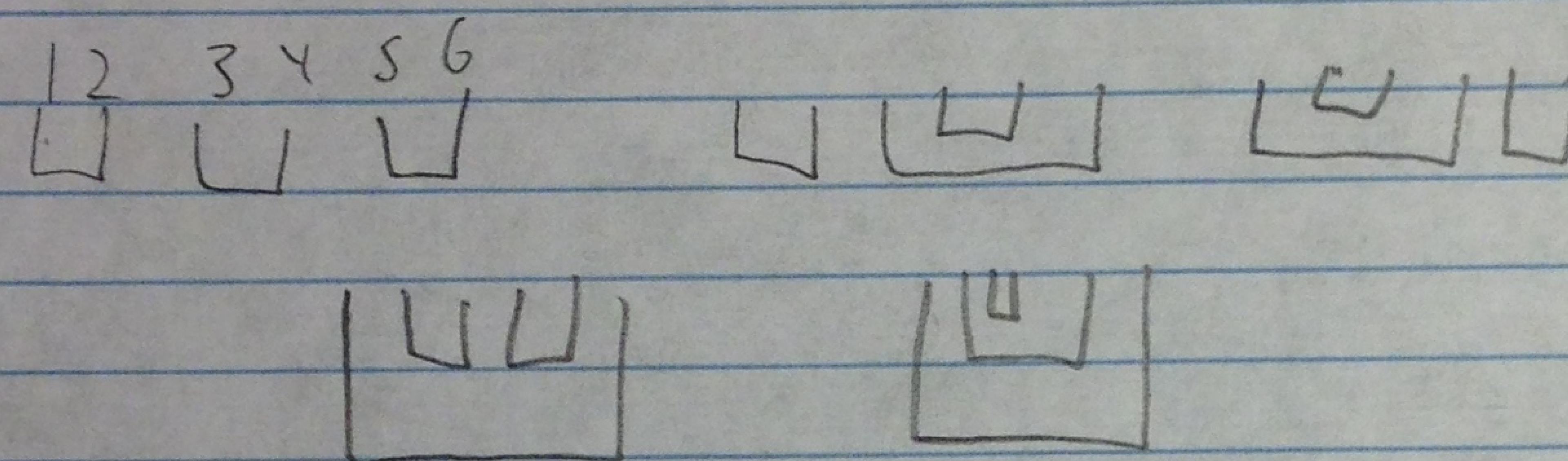


$\le$



$\pi \qquad\qquad \sigma$

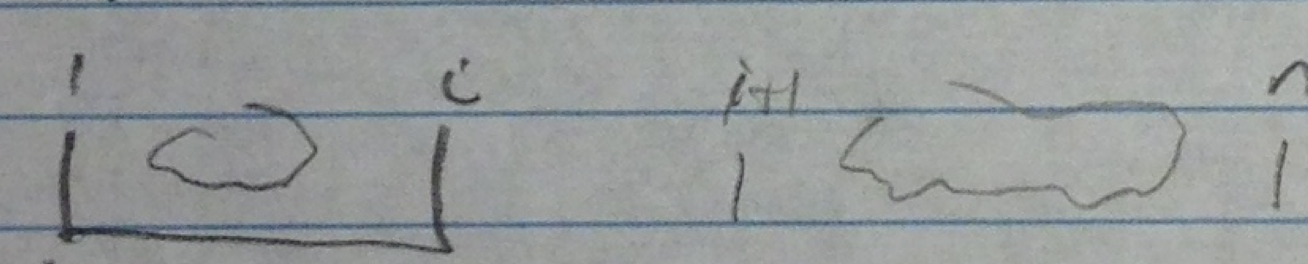
Non-crossing Pair Partitions



Catalan's original parenthesis problem $\quad$ # on $2n = C_n$

$NC^{(i)}(n)$ = non-crossing partitions where the block containing 1 has i as its largest element

$$NC^{(i)}(n) \cong NC^{(i)}(i) \times NC(n-i)$$



$$NC^{(i)}(n) \qquad NC(i-1) \times NC(n-i)$$

join $\qquad\qquad$ meet

$$\pi \vee \sigma \le \pi, \sigma \le \pi \wedge \sigma$$

The blocks of π partition n

non-crossing partitions with $1^{r_1} 2^{r_2} \dots n^{r_n}$ where $r_1 + 2r_2 + 3r_3 \dots = n$ is

$$\frac{n!}{r_1! \cdots r_n! \, (n+1-(r_1+\cdots+r_n))!}$$

Zeta for $NC(n)$ is $C_n \times C_n$

$$\text{Zeta}(\pi,\sigma) = \begin{cases} 1 & \text{if } \pi \le \sigma \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mu} = \text{Zeta}^{-1}$$

$$\langle A_1, \dots, A_4 \rangle_\pi = \frac{1}{n} E \, \mathrm{Tr}\left[(A_1 A_4)(A_2 A_3)\right]$$

$A_i = k \times k$

$$\left[\kappa_\pi(A_1, \dots, A_n) \right] = \left[\langle A_1, \dots, A_n \rangle_\pi \right] \cdot \text{Mu}$$

free cumulants

row vector of all "non-commutative" moments

$$\kappa_n(A) = \kappa_{\cancel{n}}(A, \dots, A)$$

We say A and B are free if for all free cumulants of $A + B$ and

$$\kappa_n(A+B) = \kappa_n(A) + \kappa_n(B) \qquad n = 1, 2, \dots$$

in analog of classical cumulants adding for independent random variables

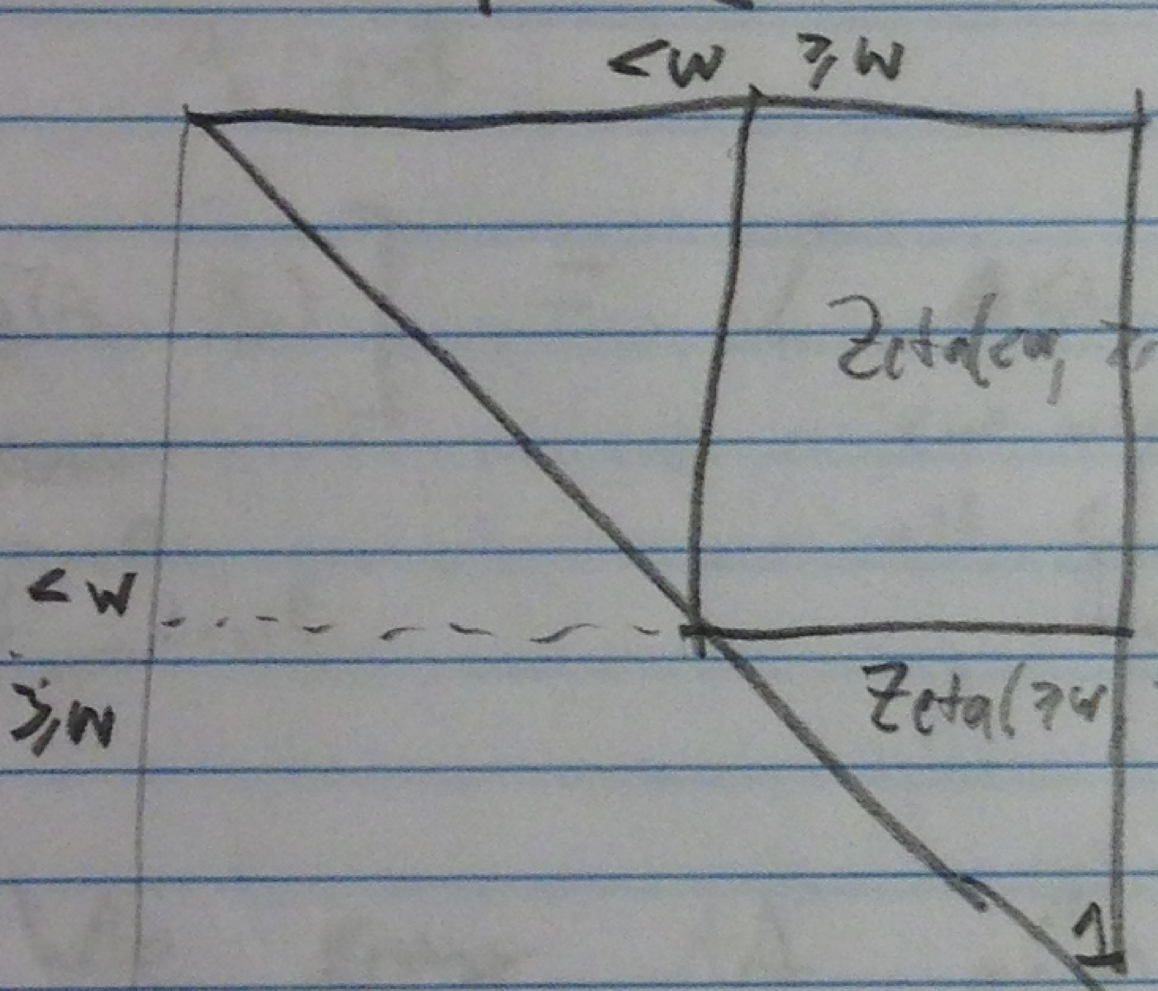
If blocks of π form partition of n $\quad 1^{r_1} 2^{r_2} \ldots n^{r_n}$

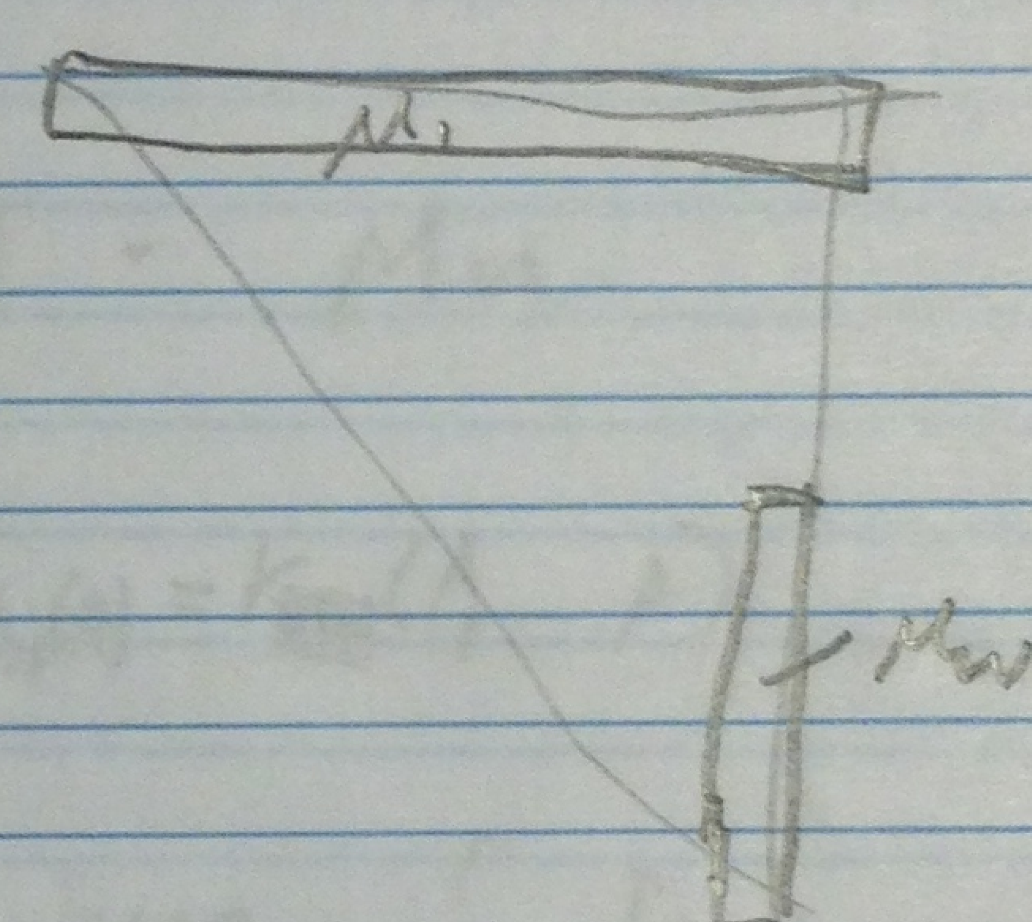
then $\quad \mu(0_n, \pi) = \prod \left[(-1)^{j-1} C_{j-1} \right]^{r_j}$

Proof based on showing $\mu(0_n, 1_n) = (-1)^{n-1} C_{n-1}$ and induction.

Cute approach:

Block Decompose Zeta + Pick a piece of Mu





$\cancel{M_w} = M_1 = Mu(0_n, :)$

$M_w = Mu(\geq w, 1_n)$

$$Zeta(\geq w, \geq w) \cdot M_w = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$M_1 * Zeta(:, \geq w) = [0 \ldots 0]$$

Now the cool part:

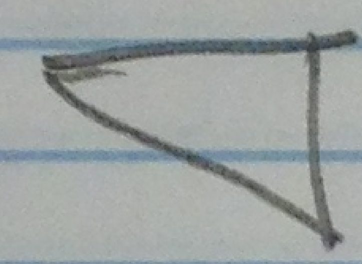
Every row in Zeta(<w, ≥w)

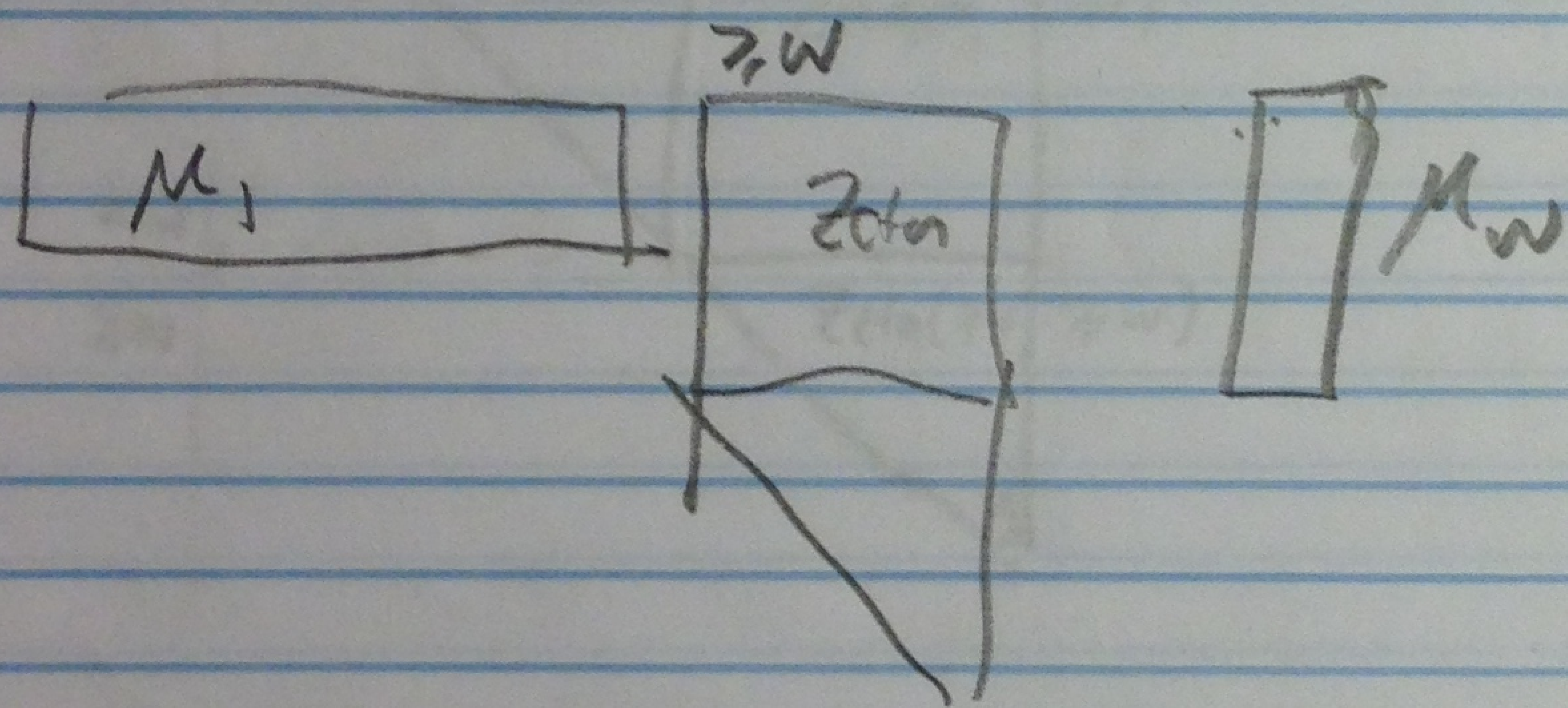
can be found in Zeta(≥w, ≥w)

Proof:
Consider $\mathrm{Zeta}(\sigma, \mu)$ for $\mu \geq w$

$$\mathrm{Zeta}(\sigma, \mu) = \begin{cases} 1 & \text{if } \mu \geq \sigma \wedge w \\ 0 & \text{otherwise} \end{cases}$$

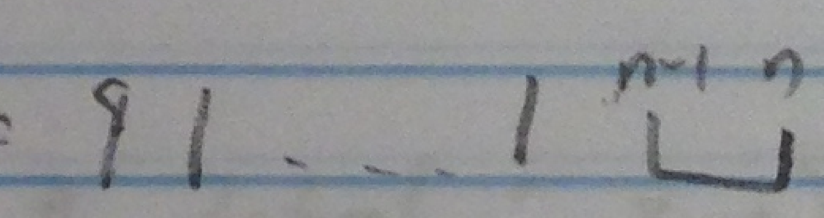
that is $\mathrm{Zeta}(\sigma, \geq w) = \mathrm{Zeta}(\sigma \wedge w, \geq w)$
which is a row in

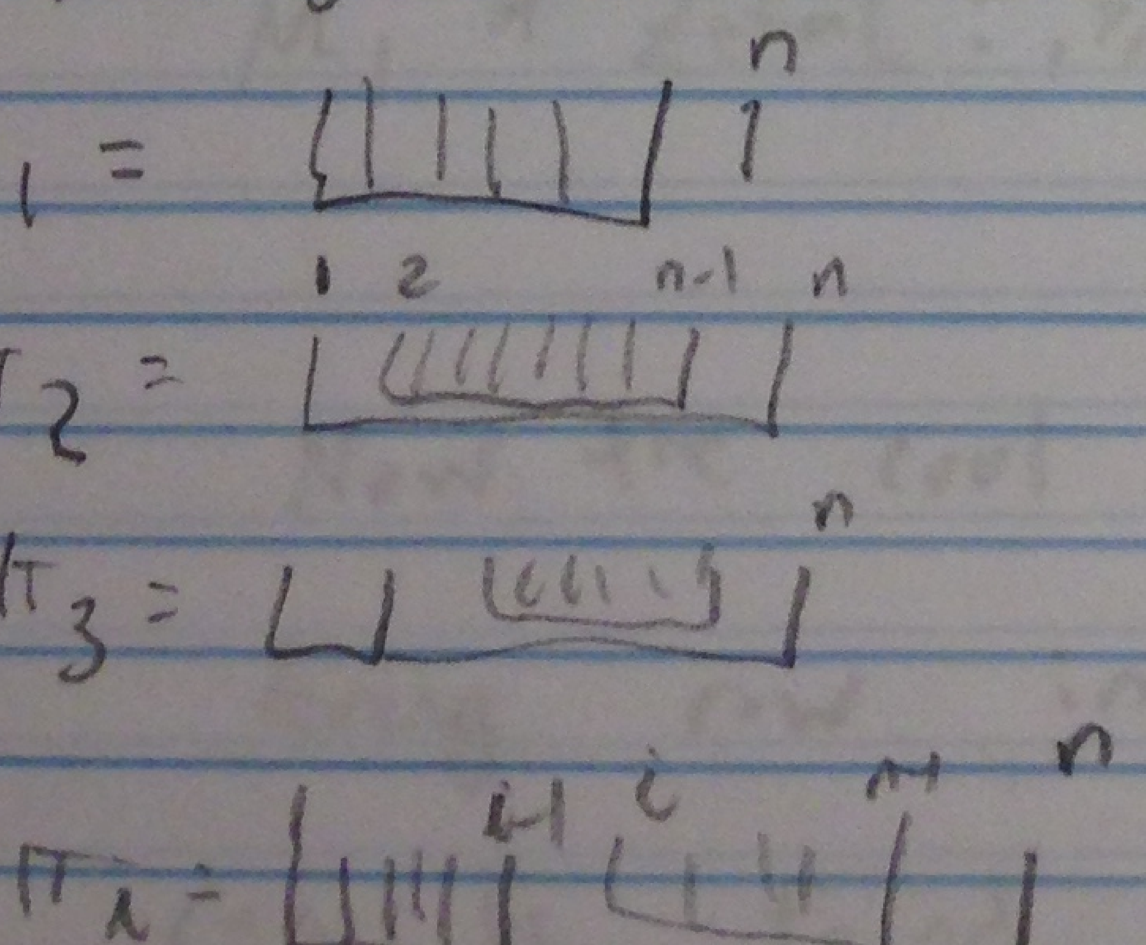




$$= 0 = \sum_{\pi \wedge w = 1_n} \mu(0_n, \pi)$$

front
Example: $w =$





position of π_i = $i + (n-i)$

& leads to

$$C_{n-1} = \sum_{i=1}^{n-1} C_{i-1} C_{n-i-1}$$