Empirical Density:

Take a random eigenvalue (uniformly take 1 from the n) from the GUE. The density is usually written (perhaps without $1/n$)

$$f(x) = \frac{1}{n}\sum_{j=0}^{n-1} \phi_j^2(x),$$

where $\phi_j(x) = (2^j j!)^{-\frac{1}{2}} H_j(x) \sqrt{\frac{1}{\sqrt{\pi}} e^{-x^2}}$

For n large $f(x)$ is nearly a semi-circle with radius $2n$.

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_j(x) H_k(x) e^{-x^2}\, dx = 2^j j!\, \delta_{jk}$$

<u>Normalization Comments</u>

The Hermite Polynomial having weight $\frac{1}{\sqrt{\pi}} e^{-x^2}$ rather than $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is denoted the physicists' Hermite polynomial. It is the one available in Mathematica/Maple & seems to be the official one. It has leading coefficient 2^n.

Here $\int_{\mathbb{R}} \phi_j(x)\, \phi_k(x)\, dx = \delta_{jk}$

The above formula for $f(x)$ is for the joint density

$$c \prod |x_i - x_j|^2 e^{-\sum x_i^2}$$

Indeed $f(x) = c \int_{x_2 \ldots x_n} \prod |x_i - x_j|^2 e^{-\sum x_i^2}\, dx_2 \ldots dx_n$

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or $$f(x) = \int_{\mathbb{R}^n} \delta(x - x_1) \quad P(x_1, \dots, x_n) \, d\cancel{x}$$

$$= \int_{\mathbb{R}^n} \left[\frac{1}{n} \sum \delta(x - x_i)\right] P(x_1, \dots, x_n) \, dx$$

In general take $w(x) \geq 0$ $\int_I w(x) = 1$

define $P(x_1, \dots, x_n) = c \prod_{i<j} |x_i - x_j|^2 \prod_{i=1}^{n} w(x_i)$ to be a prob density

Theorem: $$\int_{\mathbb{R}^n} \left[\frac{1}{n} \sum \delta(x - x_i)\right] P(x_1, \dots, x_n) \, dx = \frac{1}{n} \sum_{j=0}^{n-1} \phi_j^2(x)$$

where $\phi_j(x) = p_j(x) \sqrt{w(x)}$, $p_j(x) =$ jth orth polynomial

Cauchy - Binet

If $C = AB$

$$C\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_p \end{pmatrix} = \sum_{k_1<k_2<\cdots<k_p} A\begin{pmatrix} i_1 \cdots i_p \\ k_1 \cdots k_p \end{pmatrix} B\begin{pmatrix} k_1 \cdots k_p \\ j_1 \cdots j_p \end{pmatrix}$$

↑ det of $p \times p$ submatrix

$C = A^T D A$ A has p columns

$$\det(A^T D A) = \sum_{i_1<i_2<\cdots<i_p} A\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p \end{pmatrix}^2 d_{i_1} d_{i_2} \cdots d_{i_p}$$

A can be ~~...~~ countably infinite $\times p$

" " " continuously infinite $\times p$

$$A(x, j) \; \cancel{A_{xj}} = \phi_j(x) \qquad x \in I, \; j = 0, \ldots, p-1$$

e.g. $\cancel{A} \; C = A^T B$ $\quad A : I \times p \quad \cancel{A_{xj}} \; A(x,j) = \phi_j(x)$

$\quad B : I \times p \quad B(x,k) = \psi_k(x)$

$$\begin{array}{c c} & \text{row} \\ A = & x \rightarrow \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \phi_0(x) & \phi_1(x) & \cdots & \phi_{p-1}(x) \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \\ & \quad \text{col: } 0 \qquad\qquad\qquad p-1 \end{array}$$

$$B = \begin{bmatrix} \psi_0(x) & \psi_1(x) & \cdots & \psi_{p-1}(x) \end{bmatrix}$$

$$C_{ij} = (A^T B)_{ij} = (\phi_i(x), \psi_j(x)) = \int \phi_i(x)\, \psi_j(x)\, dx$$

$$C = \begin{pmatrix} \int \phi_0(x)\psi_0(x)\,dx & \cdots & \int \phi_0(x)\psi_{p-1}(x)\,dx \\ \vdots & \ddots & \\ \int \phi_{p-1}(x)\psi_0(x)\,dx & & \int \phi_p(x)\,\psi_{p-1}(x)\,dx \end{pmatrix}$$

$$\det C = \int_{x_1 < x_2 < \cdots < x_p} \det\left(\phi_i(x_k)\right)_{\substack{i=0,\ldots,p-1 \\ k=x_1,\ldots,x_p}} \det\left(\psi_i(x_k)\right)_{\substack{i=0,\ldots,p-1 \\ k=1,\ldots,p}} dx_1 \ldots dx_p$$

can be written $\frac{1}{p!} \int_{x_1,\ldots,x_p} \det \phi_i(x_k) \det(\psi_i(x_k))\, dx_1 \ldots dx_p$

(1883 Andréief)

A "p-fold" multivariate integral is a det of a $p \times p$ matrix of univariate integrals

Vandermonde Matrices

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{p-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_p & x_p^2 & & x_p^{p-1} \end{vmatrix} = \prod_{i<j} (x_j - x_i) = C \begin{vmatrix} p_0(x_1) & p_1(x_1) & \cdots & p_{p-1}(x_1) \\ & & & \\ p_0(x_p) & p_1(x_p) & & p_{p-1}(x_p) \end{vmatrix}$$

Application to $\beta = 2$ random matrix integrals

For no reason other than convenience consider

$$\int_{x_1,\ldots,x_n} \prod_{i=1}^{n} (1+f(x_i)) \prod_{i<j} |x_i - x_j|^2 \prod w(x_i) \, dx_1 \ldots dx_n$$

$$A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \phi_0 & \phi_1 & \cdots & \phi_{n-1} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \qquad D = \begin{bmatrix} \ddots & & \\ & 1+f(x) & \\ & & \ddots \end{bmatrix}$$

$$\cancel{B = \begin{bmatrix} \uparrow & \uparrow & & \\ \phi_0 & \phi_1 & & \phi_{n-1} \\ \downarrow & \downarrow & & \end{bmatrix}}$$

$$C_{ij} = \int \phi_i(x)\, \phi_j(x) \left(1 + f(x)\right) dx$$

$$= \delta_{ij} + \int \phi_i(x) \phi_j(x)\, f(x)\, dx$$

Clever choices of f give interesting statistics

e.g. $f(x) = z\,\delta(x-y)$

$$\Pi\left(1+f(x_i)\right) = 1 + z\sum \delta(x-x_i) + z^2 \cdot (\text{terms that will integrate to } 0)$$

rank 1 matrix

$$C = I + z\left[\ \phi_i(x)\,\phi_j(x)\ \right]$$

$$\det C = 1 + z\sum_{i=0}^{n-1} \phi_i^2(x)$$

So $\int \left(\frac{\sum \delta(x-x_i)}{n}\right) P(\ldots) = \sum_{i=0}^{n-1} \phi_i^2(x)$

Integrals Normalize to 1

For other β we have

$\beta = 1, 4$ Praffians

Generally: Can compute Moments

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} G & \sqrt{2}\chi_{n-1} & & \\ \sqrt{2}\chi_{n-1} & \ddots & \ddots & \\ & \ddots & \ddots & \sqrt{2}\chi_1 \\ & & \sqrt{2}\chi_1 & G \end{bmatrix}$$

$E(\mathrm{Tr}\, T^k)$ can be computed readily with mathem

Consider walks

Interesting Fact

$$E(\mathrm{Tr}\, T^k) = E((T^k)_{11})$$

for these matrices.

$$\int_{x_1 \le \dots \le x_n} \det(\phi_j(x_k))_{j,k=1,\dots,n}$$

$$= \mathrm{Pf}\left(\iint \mathrm{sign}(y-x)\, \phi_j(x)\, \phi_k(y)\, dx\, dy \right)$$

Gives $\beta = 1$ for Hermite

there are "erfc's" for odd n, probably ~~[illegible]~~ iff for all β