Finite Random Matrix Theory Integrals

Much of finite random matrix theory is about finding formulas for

$$\int_{x_1, x_2, \dots, x_n} F(x) \prod_{i<j} |x_i - x_j|^\beta \prod_{i=1}^{n} w(x_i)\, (dx)^n,$$

where * $F(x_1, \dots, x_n) = F(x)$ is usually a symmetric function of x but not always

* $w(x)$ is a "weight function" or probability density

Answers come in several flavors:

1) <u>Mehta world</u>: $\beta = 2$ "determinental processes"
Exact Formulas
Lead to empirical densities
largest eigenvalues
spacings
Correlation functions: Empirical joint density of $\binom{n}{k}$ eigenvalues chosen uniformly at random

* Clever choices for F
* Consequence of Cauchy-Binet

<u>$\beta = 1, 4$</u>

There is a Cauchy-Binet variant involving pfaffians that have been mined for all their worth (probably)

$Pf(A) = \sqrt{\det(A)}$ defined for anti-sym A of even dimensions

Hermite may be more complicated or simply hasn't been investigated properly

Okounkov has the from

$$E(C_\kappa(A)) = \text{coef of Zonal in } P_{2,2,2,2,\dots 2} = \left(\sum x_i^2\right)^{k/2}$$

Don't know if

$$E(C_\kappa(AB)) = \text{const} \times C_\kappa(B)$$

Back to $\beta=2$ and Mehta

There is a determinental formula usually written

$$E\left(\prod_j (1+f(x_j))\right) = \det(I + K_n f)$$

$$(K_n f)_{ij} = \int \varphi_i(x)\,\varphi_j(x)\,f(x)\,w(x)\,dx$$

e.g. ~~f(x) = z f(x)~~

$$f(y) = z\,\delta(x-y)$$

$$(K_n f)_{ij} = z\,\varphi_i(y)\,\varphi_j(y)\,w(y)$$

$$\det(I + K_n f) = \prod \quad 1 + z\sum \varphi_i^2(y)\,w(y))$$

$$\cancel{E\left(\sum \delta(x_i-y)\right)}$$

$$\text{Prob}\left(\overset{\text{an}}{\exists}\, x_i = y\right) = \sum \varphi_i^2(y)\;w(y))$$

2) $F(x) = P_\kappa(x) P_\lambda(x)$

P_κ homog multivariate orthogonal polynomial of degree κ
integral $= \delta_{\kappa\lambda}$

Most worked out: Hermite, Laguerre, Jacobi weights

3) $F(x)$ involves Jack Polynomials

Most worked out Hermite, Laguerre, Jacobi weights

or hypergeometric functions of matrix argument

Example
Hypergeometric ($\beta=1$)

$$ {}_1F_1(a; c; X) = \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \int_{0<Y<I_m} e^{\operatorname{tr} XY} |Y|^{a-\frac{m+1}{2}} |I-Y|^{c-a-(m+1)/2} (dY)^{\wedge} $$

If $X = \alpha I$ this is symmetric & Jacobi

Gives Moments of $(\sum y_i)^k$ against Jacobi

$$ \cancel{e^{\operatorname{tr} XY} = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(XY)}{k!}} $$ ~~&~~ linearity

Zonal ($\beta=1$)

$$ \underset{A}{E}\left(C_\kappa(AB) \right) = 2^k \left(\tfrac{1}{2}n\right)_\kappa C_\kappa(B) $$

A Laguerre

$B = I$ is a symmetric case

$$ E\left(C_\kappa(AB) \right) = \frac{\left(\frac{1}{2}n_1\right)_\kappa}{\frac{1}{2}(n_1+n_2)_\kappa} C_\kappa(B) $$

A Jacobi

Hermite a bit more complicated