Hermite: $A = \mathrm{randn}(n);$ $\quad \mathrm{eig}((A+A')/2)$

Laguerre: $A = \mathrm{randn}(m,n);$ $\quad \mathrm{svd}(A)$

Jacobi: $A = \mathrm{randn}(n1,p)$ $\quad \mathrm{gsvd}(A,B) \leftarrow$ cot form
$B = \mathrm{randn}(n2,p)$

<u>OR</u>

$X = \mathrm{randn}(n,p) \quad (n = n_1 + n_2)$
$[Y, \sim] = \mathrm{qr}(X, 0)$
$\mathrm{svd}(Y(1{:}n1, :)) \leftarrow$ cosine form

Complex Versions: $A = \mathrm{randn}(n) + i*\mathrm{randn}(n)$ etc
Quaternion Versions:

Want: $A = \mathrm{randn}(n) + i*\mathrm{randn}(n) + j*\mathrm{randn}(n) + k*\mathrm{randn}(n);$
Workaround: $X = \mathrm{randn}(n) + i*\mathrm{randn}(n);$
$Y = \mathrm{randn}(n) + i*\mathrm{randn}(n);$
$A = [X\ Y;\ -\bar{Y}\ \bar{X}]$ "conj"

- These are inefficient computationally
- The better way computationally is also better theoretically in so many ways
- Began a whole study of Stochastic Differential Equations still with many open problems
- Began a whole study of beta ensembles

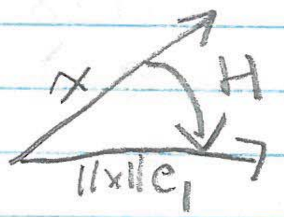
<u>Outline</u>: Numerical Linear Algebra: Sym eig problem
orthogonal invariance for chi-squared
real beta $^{(\geq 1)}$ ensembles
general beta ensembles

1. Crash course on eigenvalue computation for sym matrices
Characteristic Polynomial <u>never</u> used (too slow, too inaccurate)

$$\text{Sym Dense} \xrightarrow{\text{similar}} \text{Sym Tridiagonal} \xrightarrow{\text{similar}} \text{Diagonal}$$

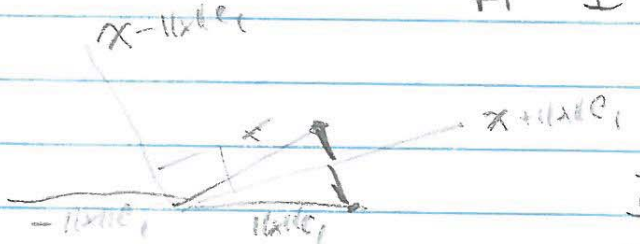
finite — infinite
slow — fast

→ eigenvalues preserved



H = Householder Reflector
$Hx = \|x\| e_1$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$H = I - \frac{2uu^T}{u^Tu}$ $\quad u = x - \|x\| e_1$



$I - \frac{uu^T}{u^Tu}$ Projects onto $x + \|x\| e_1$
$I - \frac{2uu^T}{u^Tu}$ Reflects

3

$$\begin{bmatrix} 1 & 0 \\ 0 & H(x) \end{bmatrix} \begin{bmatrix} a & x^T \\ x & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H(x) \end{bmatrix}$$

$$= \begin{bmatrix} a & \|x\| & 0 \\ \|x\| & \multicolumn{2}{c}{\multirow{2}{*}{A'}} \\ 0 & & \end{bmatrix}$$

Recur on A' to get

2) $Q * \text{randn}(n,1) \equiv \text{randn}(n,1)$ for any fixed Q

$\text{norm}(\text{randn}(n,1)) \equiv \chi_n$

Example: $[\sim, R] = \text{qr}(\text{randn}(n))$

$x = \text{randn}(n,1)$

$H(x) \cdot x = \begin{bmatrix} \chi_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\begin{bmatrix} G & G & G & G \\ G & G & G & G \\ G & G & G & G \\ G & G & G & G \end{bmatrix} \rightarrow \begin{bmatrix} \chi_4 & G & G & G \\ 0 & G & G & G \\ 0 & G & G & G \\ 0 & G & G & G \end{bmatrix} \text{ etc}$$

$$|A| \sim \chi_n \chi_{n-1} \cdots \chi_1$$

$$E|A|^2 = n! \quad (\text{True more generally})$$

Real Beta Ensembles

Tridiagonalize $\left(\frac{A+A^T}{2}\right)$

diag : var = 1

off diag : var = $\frac{1}{2}$

diag : randn(n, 1)

off-diag : $\frac{1}{\sqrt{2}}[\chi_{n-1}, \chi_{n-2}, \cdots, \chi_1]$

$$H_\beta \sim \frac{1}{\sqrt{2}}\begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & \\ \chi_{(n-1)\beta} & N(0,2) & \ddots & \\ & \ddots & N(0,2) & \\ & & \ddots & \ddots \end{pmatrix}$$

$\beta = 1$

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Theorem: The joint eigenvalue density of H_1 is the same as the GOE, GUE, GSE namely

$$c_\beta \prod |\lambda_i - \lambda_j|^\beta e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \quad \text{when } \beta = 1, 2, 4$$

$$c_\beta = (2\pi)^{-n/2} \prod_{j=1}^n \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2}j)}$$

These joint densities are already known & remain unchanged.

Theorem: For all $\beta > 0$, the joint density is correct.

Proof:

Lemma: $T = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}$

$$Q^T T Q = \Lambda \qquad \lambda_1, \dots, \lambda_n$$

$$q = Q(1, :) \qquad + q$$

Proof. To obtain the Jacobian, we will study the transformation from GOE to 1-Hermite ensemble (see Figure 2). Note that J does *not* depend on β; hence computing the Jacobian for this case is sufficient.

Let T be a 1-Hermite matrix. We know from Section II.A that the eigenvalues of T are distributed as the eigenvalues of a symmetric GOE matrix A, from which T can be obtained via tridiagonal reduction ($T = HAH^T$ for some orthogonal H, which is the product of the consecutive reflections described in Section II.A).

The joint element distribution for the matrix T is

$$\mu(a,b) \;=\; c_{a,b}\, e^{-\frac{1}{2}\sum\limits_{i=1}^{n} a_i^2} \prod_{i=1}^{n} b_i^{i-1} e^{-\sum\limits_{i=1}^{n} b_i^2}, \qquad \text{where} \qquad c_{a,b} = \frac{2^{n-1}}{(2\pi)^{n/2}\prod\limits_{i=1}^{n-1}\Gamma(\frac{i}{2})}.$$

Let

$$da = \wedge_{i=1}^{n} da_i, \quad db = \wedge_{i=1}^{n-1} db_i, \quad d\lambda = \wedge_{i=1}^{n}\lambda_i,$$

and dq be the surface element of the n-dimensional sphere. Let $\mu(a(q,\lambda), b(q,\lambda))$ be the expression for $\mu(a,b)$ in the new variables q, λ. We have that

$$\mu\,(a,b)\;da\;db \;=\; J\;\mu\,(a(q,\lambda), b(q,\lambda))\;dq\;d\lambda \;\equiv\; \nu(q,\lambda)\;dq\;d\lambda. \tag{17}$$

We combine Properties 1 and 2 of Section II.A to get the joint p.d.f. $\nu(q,\lambda)$ of the eigenvalues and first eigenvector row of a GOE matrix, and rewrite it as

$$\nu(q,\lambda)\;dq\;d\lambda \;=\; n!\;c_H^1\;\frac{2^{n-1}\Gamma(\frac{n}{2})}{\pi^{n/2}}\;\Delta(\lambda)\;e^{-\frac{1}{2}\sum\limits_i \lambda_i^2}\;dq\;d\lambda.$$

We have introduced the $n!$ and removed the absolute value from the Vandermonde, because the eigenvalues are ordered. We have also included the distribution of q (as mentioned in Property 2, it is uniform, but only on the all-positive 2^{-n}th of the sphere because of the condition $q_i \geq 0$).

Since orthogonal transformations do not change the Frobenius norm $||A||_F = \sum\limits_{i,j=1}^{n} a_{ij}^2$ of a matrix A, from (17), it follows that

$$J \;=\; \frac{\nu(q,\lambda)}{\mu(a,b)} = \frac{n!\;c_H^1\,\frac{2^{n-1}\Gamma(\frac{n}{2})}{\pi^{n/2}}}{c_{a,b}}\;\frac{\Delta(\lambda)}{\prod\limits_{i=1}^{n} b_i^{i-1}}.$$

All constants cancel, and by Lemma 2.7 we obtain

$$J \;=\; \frac{\prod\limits_{i=1}^{n-1} b_i}{\prod\limits_{i=1}^{n} q_i}.$$

Note that we have not expressed $\mu(a,b)$ in terms of q and λ in the above, and have thus obtained the expression for the Jacobian neither in the variables q and λ, nor a and b, solely; but rather in a mixture of the two sets of variables. The reason for this is that of simplicity. $\square$