lec 3 2/15/2012

Joint Element Densities

Last time: (Hermite)

If $S = \frac{A+A^T}{2}$, where $A = \text{randn}(n)$

(Normalization: $\sqrt{\frac{n}{2}} \times$ previous one)

then the joint element density is

$$2^{-n/2} \pi^{-n(n+1)/4} e^{-\frac{1}{2}\|S\|_F^2} (dS)^\wedge$$

where $(dS)^\wedge$ is my fancy way of denoting $(dS_{11}\, dS_{22} \cdots dS_{nn})(dS_{12}\, dS_{23} \cdots)$ or $(dS)^\wedge = \prod_{i \le j} dS_{ij}$, a natural volume element for symmetric matrices.

This joint density is the product of the n independent diagonals with variance $= 1$ and the $n(n-1)/2$ upper triangles with variance $= 1/2$.

In chapter (9) we will see that

$$(dS)^\wedge = \prod_{i<j} |\lambda_i - \lambda_j| \, (d\Lambda)^\wedge (Q^T dQ)^\wedge$$

describes the Jacobian from symmetric matrices to eigenvalues & eigenvectors.

For now let it suffice that integrating out the eigenvectors we transform a joint element density $f(S)$ to a joint eigenvalue density by writing

$$\frac{\pi^{n^2/2}}{\Gamma_n(n/2)} f(\Lambda) \prod_{i<j} (\lambda_i - \lambda_j) \qquad \lambda_1 \ge \cdots \ge \lambda_n$$

Summary for Hermite

GOE joint eigenvalue density is

$$\frac{2^{-n/2}}{\prod_{i=1}^{n}\Gamma(\frac{i}{2})} e^{-\frac{1}{2}\sum \lambda_i^2} \prod_{i<j}(\lambda_i - \lambda_j)\ d\lambda_1 \cdots d\lambda_n$$

$$(\lambda_1 < \lambda_2 < \cdots < \lambda_n)$$

Laguerre

Let $A = \text{randn}(m, n) \quad m \geq n$

Let $W = A^T A \quad (n \times n)$

Joint element density of A is

$$(2\pi)^{-mn/2} e^{-\frac{1}{2}\|A\|_F^2} (dA)^\wedge$$

In Theorem (9.2) we will learn that the Jacobian of a QR factorization of A is

$$(dA)^\wedge = \prod_{i=1}^{n} r_{ii}^{m-i} (Q^T dY)^\wedge (dR)^\wedge$$

where Q is $m \times m$, $A = QR$, + Y = 1st n columns of Q.

Also if (Cholesky)

$W = R^T R$ then

$$(dW)^\wedge = 2^n \prod_{i=1}^{n} r_{ii}^{n+1-i} (dR)^\wedge$$

Putting these two together and integrating out $(Q^T dQ)^\wedge$ we get

The joint density of the elements of a Wishart matrix are

$$\cancel{\prod_i \lambda_i^{(m-n-1)} \prod (\lambda_i - \lambda}$$

$$\frac{1}{2^{mn/2}\Gamma_n(m/2)} e^{-\frac{1}{2}\operatorname{tr} W} (\det W)^{(m-n-1)/2} (dW)^\wedge$$

$$\Gamma_n(m/2) = \pi^{n(n-1)/4} \prod_{i=1}^{n} \Gamma\left(\frac{m}{2} - \frac{i-1}{2}\right)$$

Question: If $m=n+1$, are the elements independent? Answer: no. While it looks like the off-diagonals are uniform & the diagonals are exponential, It's not true. There is also the constraint of W being positive definite.

Wishart joint eigenvalue density:

$$\frac{\pi^{n^2}}{\Gamma_n(n/2)\Gamma_n(m/2)} e^{-\frac{1}{2}\sum \lambda_i} \prod \lambda_i^{(m-n-1)/2} \prod_{i<j} (\lambda_i - \lambda_j)$$

$$(\lambda_1 < \lambda_2 < \ldots < \lambda_n)$$