Lec 2 2/13 (1)

Eigenvalue Joint Densities of Classical Ensembles

Prob Density Function:

univariate:

$$P(a \le x \le b) = \frac{\# \text{ in } [a,b]}{\text{total}} = \int_a^b f(t)\,dt \approx F(a)\Delta t$$

bivariate

$$P(a \le x \le b,\ c \le y \le d) = \frac{\# \text{ with } x \in [a,b],\ y \in [c,d]}{\text{total}} = \int_{u=c}^{d}\int_{t=a}^{b} P(t,u)\,dt\,du$$

Recall Classical Ensembles (real case)

Hermite GOE $g = \text{randn}(n)$; $GOE(n)$: $(g + g')/\sqrt{2n}$

Laguerre WISHART $g = \text{randn}(m,n)$; $\text{Wishart}(n,m)$: $(g' * g)/m$ $(m \ge n)$

Jacobi MANOVA $w1 = \text{wishart}(n, m1)$; $w2 = \text{wishart}(n, m2)$; $\text{Manova}(n, m_1, m_2)$: $(W_1 + W_2) \backslash W1$

Density Laws from infinite RMT: So far you've seen a semicircle Proof in the notes

Hermite: Semicircle $\frac{1}{2\pi}\sqrt{4-t^2}$

Laguerre: Marcenko-Pastur $\frac{\sqrt{(x-b_-)(b_+-x)}}{2\pi x c}$ $c = n/m \le 1$, $b_\pm = (1 \pm \sqrt{c})^2$

Jacobi: In the notes

Derivation of Joint Element Density of GOE

Easiest might be $\frac{g+g'}{2} = \text{GOE} \times \sqrt{n/2}$

For $(g+g')/2$:

Diagonal: Just randn() Mean 0 Variance 1
Off Diagonal: Mean 0 Variance $\frac{1}{2} = \left(\frac{1+1}{4}\right)$

Joint Density: Just Multiply (Note: let's call this S)

Diagonal: $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
off-diagonal: $\frac{1}{\sqrt{\pi}} e^{-x^2}$

$$\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^n a_{ii}^2/2} \left(\frac{1}{\sqrt{\pi}}\right)^{\frac{n(n-1)}{2}} e^{-\sum_{i<j} a_{ij}^2}$$

$$= 2^{-\frac{n(n+1)}{2}} \pi^{-\frac{n(n-1)}{2}} e^{-\frac{1}{2}\sum_{1\le i,j\le n} a_{ij}^2/2}$$

counts off diagonals twice

$$= c\, e^{-\frac{1}{2}\|A\|_F^2}$$

To emphasize it is a joint prob density:

$$F(A) = c e^{-\frac{1}{2}\|A\|_F^2} \prod_{i \le j} da_{ij}$$

or $$c e^{-\frac{1}{2}\|A\|_F^2} \bigwedge_{i \le j} da_{ij}$$

Don't be frightened: just ordinary volume in $\frac{n(n+1)}{2}$ space

or my own notation

$$c e^{-\frac{1}{2}\|A\|_F^2} (dA)^{\wedge}$$

Deriving Joint Eigenvalue Density

1. The quick sloppy way
2. The careful way

The quick sloppy way: Ignore constants + Trust that Eigenvectors Don't Matter + the Jacobian is essentially

$$(dA)^{\wedge} = \prod_{i<j} |\lambda_i - \lambda_j|$$

Careful Way

Not much Different

Example $n=2$:

See Example 8 of the Jacobian Eigenvalue Section

$$S = \begin{bmatrix} p & s \\ s & r \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T$$

$$J = \begin{bmatrix} \frac{\partial p}{\partial \theta} & \frac{\partial p}{\partial \lambda_1} & \frac{\partial p}{\partial \lambda_2} \\ \frac{\partial r}{\partial \theta} & \frac{\partial r}{\partial \lambda_1} & \frac{\partial r}{\partial \lambda_2} \\ \frac{\partial s}{\partial \theta} & \frac{\partial s}{\partial \lambda_1} & \frac{\partial s}{\partial \lambda_2} \end{bmatrix}$$

$$\det(J) = |\lambda_1 - \lambda_2|$$

Thus to change to eig variables

$$e^{\frac{1}{2}\|S\|_F^2} \, dp \, dr \, ds$$

$$= e^{-\frac{1}{2}(\lambda_1^2+\lambda_2^2)} |\lambda_1 - \lambda_2| \, d\lambda_1 \, d\lambda_2 \, d\theta$$

<u>In general:</u>

$$e^{\frac{1}{2}\|S\|_F^2} (dS)^{\wedge} = e^{-\frac{1}{2}\sum \lambda_i^2} \prod |\lambda_i - \lambda_j| \underset{\substack{\uparrow \\ \text{Eigenvector factors}}}{} d\lambda_1 \, d\lambda_2 \ldots d\lambda_n$$

~~Interpretation:~~

General Theorem:

If $n \times n$ S is invariant under orthogonality transformations then the joint density of $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ is

$$\frac{\pi^{n^2/2}}{\Gamma_n(n/2)} f(\Lambda) \prod_{i<j} |\lambda_i - \lambda_j|$$

where $\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a - \frac{(i-1)}{2}\right)$

Interpetation:

Repulsion

$$\frac{1}{2^{n/2} \prod_{i=1}^{n} \Gamma(i/2)} e^{-\frac{1}{2}\sum \lambda_i^2} \prod_{i<j} \lambda_i - \lambda_j$$

Things to Come

Many Results Require integration over orthogonal matrices (fancy analog of integration over circle)

Zonal Polynomial A First View

Let $A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$ $\quad Q =$ Random

$B = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}$

$$\underset{Q}{E}\left(e^{\operatorname{tr} AQBQ^T} \right) = \text{Polynomial in } A \text{ \& } B$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{C_\kappa(A) C_\kappa(B)}{k!\, C_\kappa(I)}$$

For the unitary case we get Schur polynomials &

Harish-Chandra-Itzykson-Zuber-Integral