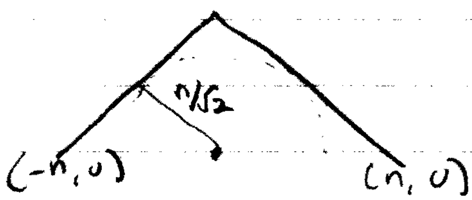
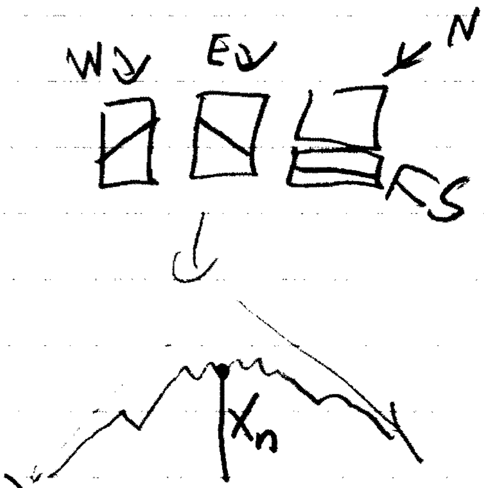
Aztec Diamond Fluctuation Formula





$$\lim_{n \to \infty} P\left(\frac{X_n - \frac{n}{\sqrt{2}}}{2^{-5/6} n^{1/3}} \le s \right) = F_2(s)$$

More can be said, but this is a good feel.

Cuernavaca Bus Model

A Jump process may be simulated by sort(rand(n,1))

Example:

```
n=100;
stairs([0 sort(rand(1,n)) 1], (0:(n+1))/(n+1))
```

Schur Polynomials

$$s_\lambda(x_1,\dots,x_n) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & & x_n^{\lambda_2+n-2} \\ \vdots & & \\ x_1^{\lambda_n} & & x_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & & x_n^{n-1} \\ x_1^{n-2} & & \\ & & \\ x_1^{0} & & x_n^{0} \end{vmatrix}}$$

$$s_\lambda(\underset{\substack{\uparrow \\ \text{matrix}}}{X}) = s_\lambda(\operatorname{eig}(X)) \quad \text{if } X \text{ is an } n{\times}n \text{ matrix}$$

well defined since s_λ is symmetric in $x_1,\dots,x_n$

Relation to Jack Polynomials

$s_\lambda = J_\lambda^{(1)}$ $(\alpha=1)$ with leading coef 1

In MOPS `jack(1, [2,1,1], [x,y,z], 'P')` computes $s_{[2,1,1]}(x,y,z)$

Relation to Power Functions (Traces)

$$p_k(x) = \sum_{i=1}^{n} x_i^k \quad = \mathrm{Tr}\, X^k$$

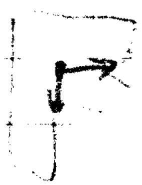
$$p_\lambda(x) = \prod_i p_{\lambda_i}(x) = (\mathrm{Tr}\, X^{\lambda_1})(\mathrm{Tr}\, X^{\lambda_2}) \cdots$$

In general, the character table connects Schur to Powers.
Specifically

$$p_{1^n}(x_1, \ldots, x_\ell) = \left[\mathrm{Tr}(X_{\ell \times \ell})\right]^n$$

$$= \sum_{\substack{|\lambda| = \ell \\ \lambda \vdash n}} d_\lambda \, S_\lambda(X)$$

where $d_\lambda = \dfrac{n!}{\prod_c h_c}$ = # of Young Tableau of shape λ

 h_c = # below + # right + c

(more soon)
on page (5)

Orthogonality & Eigenfunction Property

(1) $E_U \, s_\lambda(U)\overline{s_\kappa(U)} = \delta_{\lambda\kappa}$

unitary Haar measure

(2) $E_U \, s_\lambda(U^* A U B) = \dfrac{s_\lambda(A)s_\lambda(B)}{s_\lambda(I)}$

Note: For $\beta \neq 2$, (1) refers to the circular ensembles and (2) is what I call ghost Haar

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distributed. It is not hard to construct a proof along these lines. Next consider

$$\int_{U_n} (\mathrm{Tr}(m))^a (\overline{\mathrm{Tr}(m)})^b \, dm.$$

This has a group-theoretic interpretation: $(\mathrm{Tr}(m))^a$ is the character of the ath tensor power of the n-dimensional representation of U_n. The integral is the sum of the multiplicities of the common constituents of the ath and bth tensor powers. In particular it is an integer. By the first remark it converges to $\boldsymbol{E}(Z^a \bar{Z}^b)$ with Z complex normal. These last moments are integers as well. Since integers converging to integers must eventually be equal, we expect equality of moments in all the cases of this paper. It is interesting how rapidly this takes hold.

Hint of β-circular vs β-Haar meeting up at $\beta = 2$ (ghost)

Remark. The physics literature works with a unitary, orthogonal and symplectic ensemble. While the unitary ensemble is the one considered here, the orthogonal and symplectic ensembles differ. Their orthogonal ensemble consists of the symmetric unitary matrices. This is U_n/O_n. Their symplectic ensemble consists of anti-symmetric unitary matrices. This is U_{2n}/Sp_n. We hope to carry through the distribution of the eigenvalues on these ensembles along the lines of the present paper.

1. The unitary group

A complex normal random variable Z can be represented as $Z = X + iY$ with X and Y independent real normal random variables having mean 0 and variance $\frac{1}{2}$. These variables can be used to represent Haar measure on the unitary group U_n in the following standard fashion. Form an $n \times n$ random matrix with independent identically distributed complex normal coordinates Z_{ij}. Then perform the Gram–Schmidt algorithm. This results in a random unitary matrix M which is Haar distributed on U_n. Invariance of M is easy to see from the invariance of the complex normal vectors under U_n.

This representation suggests that there is a close relationship between the unitary group and the complex normal distribution. For example, Diaconis and Mallows (1986) proved the following result.

Theorem 0. *Let M be Haar distributed on U_n. Let Z be complex normal. Then, for any open ball B,*

$$\lim_{n\to\infty} \boldsymbol{P}\{\mathrm{Tr}M \in B\} = \boldsymbol{P}\{Z \in B\}.$$

The following result generalizes Theorem 0.

Theorem 1. *Fix k in $\{1, 2, 3, \ldots\}$. For every collection of open balls B_i in the complex plane,*

$$\lim_{n\to\infty} \boldsymbol{P}\{\mathrm{Tr}(M) \in B_1, \mathrm{Tr}(M^2) \in B_2, \ldots, \mathrm{Tr}(M^k) \in B_k\} = \prod_{j=1}^{k} \boldsymbol{P}(\sqrt{j}Z \in B_j).$$

Diaconis Shahshahani 1994

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1	2	7	10
3	5	9	
4	8		
6	11		

Figure 33.3: A Ferrers diagram for $(4,3,2,2)$ and example of a Young tableau with shape λ

pairs of Young tableaux.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a set of integers such that $\sum \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then we say λ is a *partition of* n and write $\lambda \vdash n$. We denote the number of entries in λ by $|\lambda|$, which in this case is equal to k. In the special case where $\lambda = (1, 1, \ldots, 1)$ we write $\lambda = 1^n$.

A *Ferrers diagram* with shape λ is a set of cells as shown in Figure 33.3, where row i has λ_i cells. A *standard Young tableau* is a Ferrers diagram with the numbers $1, 2, \ldots, n$ in the cells such each number is used once, and the entries increase along each row and down each column.

The *hook length* h_c of a cell c is the number of cells to the right of c in its row plus the number of cells below c in its column plus the cell c itself. Thus in the tableau in Figure 33.3 the hook length of the cell containing 5 is $h_c = 4$, and the hook length of the cell containing 8 is $h_c = 2$.

An interesting question is this: how many standard Young tableaux are there of shape λ? The answer is surprisingly simple, and given by the following theorem.

Proposition 33.2. *(Hook Formula) Let d_λ be the number of standard Young tableaux of shape λ. Then:*

$$d_\lambda = \frac{n!}{\prod_c h_c} \tag{33.1}$$

If we look at the tableau in Figure 33.3 we can calculate the number of tableau of that shape:

$$d_{(4,3,2,2)} = \frac{11!}{7 \cdot 6 \cdot 3 \cdot 1 \cdot 5 \cdot 4 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 1320$$

This is far too many to explicitly verify, so we can look at an easier example. Consider $\lambda = (3, 2) \vdash 5$. The hook length formula tells us we can only construct $120/(4 \cdot 3 \cdot 1 \cdot 2 \cdot 1) = 5$ standard Young tableaux. These are shown in Figure 33.4. It is left as an exercise to the reader to prove there exist no more standard Young tableaux of that shape.

1	2	3
4	5	

1	3	5
2	4	

1	2	5
3	4	

1	2	4
3	5	

1	3	4
2	5	

Figure 33.4: All standard Young tableaux of shape (3,2)

Unfortunately, no simple combinatorial proof of the hook formula exists. A number of outlines of existing proofs are given by Sagan ([389], p. 266).

Young tableaux are related to the permutation group by a construction called the Schensted correspondence, or perhaps more appropriately, the Robinson-Schensted-Knuth correspondence. For

Combinatorial Fact

$$\sum_{\lambda \vdash n} d_\lambda^2 = n!$$

Must count something!

$$\sum_{\substack{\lambda \vdash n \\ |\lambda| = \ell}} d_\lambda^2 = \# \pi \in S_n \text{ with } \ell(\pi) = \ell$$

d_λ = 1st column of the character table

Random Matrix Fact

$$[\mathrm{Tr}(\underbrace{U}_{\ell \times \ell})]^n = \sum_{\substack{\lambda \vdash n \\ |\lambda| \le \ell}} d_\lambda \, S_\lambda(x)$$

$$E\left(|\mathrm{Tr}(U)|^{2n} \right) = \sum_{\substack{\lambda \vdash n \\ |\lambda| \le \ell}} d_\lambda^2$$

Immediate consequence from (1) on p. 4

For every n, there is a matrix $\chi^\lambda(\mu)$ indexed by partitions, such that

$$p_\mu = \sum_\lambda \chi^\lambda(\mu) s_\lambda$$

e.g. $n=3$

$\chi^\lambda(\mu)$	μ: (1,1,1)	(2,1)	(3)
λ: (1,1,1)	1	-1	1
(2,1)	2	0	-1
(3)	1	1	1

Partitions may be denoted $1^{m_1} 2^{m_2} 3^{m_3} \ldots$
where m_k = # of times k appears in λ

e.g. $(1,1,1) = 1^3$
$(2,1) = 1^1 2^1$
$(3) = 3^1$

Let $z_\lambda = (1^{m_1} 2^{m_2} \ldots) m_1! m_2! \ldots$

so $z_{1^3} = 6$
$z_{(2,1)} = 2$
$z_3 = 1$

$$\chi^T \chi = \operatorname{diag}(z_\lambda)$$

$\Rightarrow$

$$s_\lambda = \sum_\mu \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu$$

Example:

$$p_{1,1,1}(X) = (\operatorname{Tr} X)^3$$

$$= S_{1,1,1}(x) + 2\, S_{2,1}(x) + S_3(x)$$

$$z_\lambda = n! \,/\, (\#\text{ of permutations with cycle type } \lambda)$$

$$d_\lambda = \cancel{\chi}^{\lambda}(1^n)$$

$$\sum d_\lambda^2 = n!$$

$$p_{1^n}(X) = (\operatorname{Tr} X_{\ell\times\ell})^n = \sum_{\substack{\lambda \vdash n \\ |\lambda| \le \ell}} d_\lambda S_\lambda(x)$$

$$E(\,|\operatorname{tr} X|^{2n}\,) = \sum_{\substack{\lambda \vdash n \\ |\lambda| \le n}} d_\lambda^2$$

This moment will be interpreted as counting longest increasing subsequences