Combinatorial Models that lead to Tracy Widom

1. Longest Increasing Subsequence
2. Corner Growth
3. Aztec Diamonds
4. Cuernavaca Mexico Busses

Reminder: Tracy-Widom

intimidating ODE, a few lines of MATLAB,

$$F_2(s) = e^{-\int_s^\infty (x-s)^2 q(x)\,dx}, \quad q'' = sq + 2q^3,\ q(s) \sim Ai(s) \text{ as } s \to \infty$$

$$Ai''(s) = s\,Ai(s),\ Ai(\infty) = 0.$$

0. Random Matrices

A = randn(n) + i randn(n)

S = (A+A')/2

$$\lim_{n\to\infty} P\left(n^{1/6}(\lambda_{max}(S) - 2\sqrt{n}) \le s\right) = F_2(s)$$

1. Longest Increasing Subsequence

$$\lim_{n\to\infty} P\left(\frac{l(\text{randperm}(n)) - 2\sqrt{n}}{n^{1/6}} \le s\right) = F_2(s)$$

Combinatorial Problems that lead to Tracy–Widom

$$K(x,y) = \sum_{i=0}^{n-1} \phi_i(x)\phi_i(y) \qquad \phi_i(x) = \pi_i(x)\sqrt{w(x)}$$

GUE: $\pi_i(x)$ = orthonormal Hermite

$$\lim_{\substack{n\to\infty \\ \text{Normalized properly}}} K(x,y) = \frac{\sin x \cos y - \sin y \cos x}{x-y} = \frac{\sin(x-y)}{x-y}$$

$$\Pr(\lambda_{max} \le s) = \det(I - K_s)$$

Normalization: A = randn(n) + i randn(n)
S = (A+A')/2

$$\lambda'_{max} = n^{1/6}(\lambda_{max} - 2\sqrt{n})$$

$$P(\lambda'_{max} \le s) = \det(I - A_s) = F_2(s) = e^{-\int_s^\infty (x-s)q^2(x)\,dx}$$

$$q'' = sq + 2q^3$$

$$q(s) \sim Ai(s), \quad s\to\infty$$

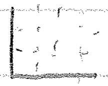
$$A(x,y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$$

$$A_s = A(x,y) \text{ on } [s,\infty)$$

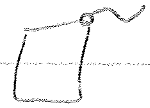
$$Ai''(x) = xAi(x)$$

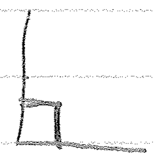
$$Ai(\infty) = 0$$

2. Corner Growth

$t=0$  $q = P(\text{drawing a square} \mid \text{left \& bottom})$

$$P(w(1,1)=K) = (1-q)q^K$$

 Square Label

$t = w(1,1)+1$ 

$\Pi_{M,N}$ = path from $(1,1)$ to (M,N) up & right

$$w(i,j) = \begin{cases} \text{geometrically dist random variable} \\ P(w=K) = (1-q)q^K = p(1-p)^K \end{cases}$$

$$G(M,N) = \max_{\pi \in \Pi_{M,N}} \sum_{(i,j)\in\pi} w(i,j)$$

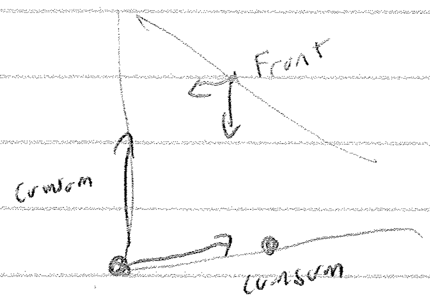
$$G^*(M,N) = \max_{\pi \in \Pi_{M,N}} \sum_{(i,j)\in\pi} w^*(i,j)$$

where $w^*(i,j) = w(i,j)+1$

$$G^*(M,N) = G(M,N) + M + N - 1$$

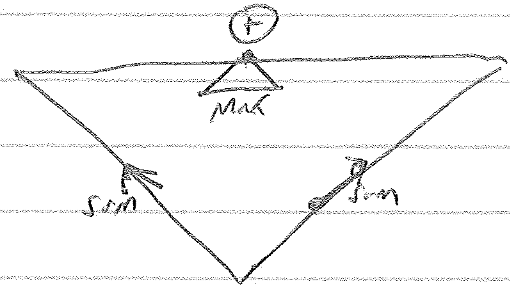
SIMULATION

w = geornd(p, M, N)



$$w(i,j) = w(i,j) + \max(w(i-1,j), w(i,j-1))$$

Triangle view:



Fronts: $y = \left(\sqrt{(1-q)(1-x)} - \sqrt{qx}\right)^2$

Ellipse Arcs tangent to the coordinate axes

Not obvious for $q > 0$

~~[illegible]~~ For $\gamma \geq 1$

$$\lim_{N\to\infty} \frac{1}{N} E(G(\lfloor \gamma N \rfloor, N)) = \frac{(1+\sqrt{q\gamma})^2}{1-q} - 1$$

$w(\gamma, q)$

$$\lim_{N\to\infty} P\left(\frac{G(\lfloor \gamma N \rfloor, N) - N w(\gamma, q)}{\sigma(\gamma, q)\, N^{1/3}} \leq s \right) = F_i^{\sigma}$$

$$\sigma(\gamma, q) = \frac{q^{1/6}\,\gamma^{-1/6}}{1-q}\left(\sqrt{\gamma}+\sqrt{q}\right)^{2/3}\left(1+\sqrt{q\gamma}\right)^{2/3}$$

Recall given $w(x)$

$$p(x_1,\dots,x_n) = c\,\Pi\,|x_i - x_j|^2\;\Pi\, w(x_i)$$

~~for~~ $w(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for GUE

Discrete w:

$$w_K^q(x) = \binom{x+K-1}{x} q^x \; : \; x = 1, 2, 3, \dots$$

Meixner weight

$$p^M(h_1,\dots,h_n) = c\,\Pi\,|h_i - h_j|^2\;\Pi\, W_K^q(h_i)$$

Can define Meixner Polynomials of Matrix Argument, etc

$$\int\cdots\int_{(-\infty,t)^n} p(x)$$

$$\sum_{\max(h) \le t+(N-1)} \cdots \sum p^M(h) = P\left(G(M,N) \le t\right)$$

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Some mathematical pieces

Ferrers Diagram
Young Tableaux
Schensted Correspondence for permutations
Knuth Correspondence for M×N matrices with nonnegative integers
Schur Polynomial
Characters of the Symmetric Group