Random Matrix Theory

* Quick Tour of the Notes
* This Semester
  - Finite Random Matrix Theory
  - Tracy Widom & Growth Processes
  - Free Probability

Finite Random Matrix Theory

* Exact formulas: entries usually standard normals
* $n=1$: Undergrad Formulas
* $n=\infty$: Important Infinite Random Matrix Theory Formulas
* n finite: More structure than $n\geq 1$
  Opportunities for research
  for applications

Scalar Random Variables ($n=1$)

Math	MATLAB	Prob Density	~~Random Vector [illegible]~~
Standard Normal	randn()	$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	
Chi-Squared χ_v^2	chi2rnd(v)	$x^{v/2-1} e^{-x/2} \frac{1}{2^{k/2}\Gamma(k/2)}$	norm(randn(v,1))^2
Beta Distribution	betarnd(A,B)	$\frac{x^{A-1}(1-x)^{B-1}}{\int_0^1 u^{A-1}(1-u)^{B-1}du} \leftarrow B(a,b)$	X=chi2rnd(2A) Y=chi2rnd(2B) X/(X+Y)

Chi-Squared With Vectors:

```
X=randn(v,1)
Return X'X
```

Beta Distribution with vectors

```
x=randn(2A,1)
y=randn(2B,1)
g=gsvd(x,y)
Return g^2/(1+g^2)
```

Random MATRICES (real case)

$g = \text{randn}(n,n)$ (nxn iid normals)
"Ginebre Matrix"

Hermite — Gaussian Ensemble (Wigner 1955) — $g=\text{randn}(n,n)$
Return $(g+g')/\sqrt{2n}$ (nxn sym)

Laguerre — Wishart Matrices (Wishart 1928) — $g=\text{randn}(n,m)$
Wishart(n,m)
Return $(g * g')/m$ (nxn sym pos)

Jacobi — MANOVA MATRICES — $W1 = \text{wishart}(n, m1)$
$W2 = \text{wishart}(n, m2)$
Return $(W1 + W2)\backslash W1$

$n=1$
Gaussian
chi-squared
Beta

(must check Free prob name)

$n \to \infty$
Semi Circle
Marcenko-Pastur "Free Poisson"
[General McKay etc]

↓

Classical orthogonal Polynomials (Warning Normalizations + Scalings vary)

Hermite Polynomials (Probabilists Form)

$H_0 = 1$
$H_1 = x$
$H_2 = x^2 - 1$
$H_3 = x^3 - 3x$
$H_4 = x^4 - 6x^2 + 3$

H_j is deg j

orthogonal with respect to a Gaussian
$G = \text{randn}()$

Pain that these are not 1

$$E(H_j(G)H_k(G)) = \delta_{jk}\, k!$$

$$\int H_j(x) H_k(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \delta_{jk}\, k!$$

Laguerre Polynomials

$L_0^{(\alpha)} = 1$
$L_1^{(\alpha)} = -x + \alpha + 1$

$L_2^{(\alpha)} = \frac{x^2}{2} - (\alpha+2)x + \frac{(\alpha+2)(\alpha+1)}{2}$

$$\int_0^\infty e^{-x} x^\alpha L_k^{(\alpha)}(x) L_j^{(\alpha)}(x)\, dx = \delta_{jk}\, k!\, \Gamma(\alpha+1+k)$$

$$E_{\chi^2_k}\left(L_j^{(\alpha)}\left(\frac{x}{2}\right) L_k^{(\alpha)}\left(\frac{x}{2}\right)\right) = \text{constant} * \delta_{jk}$$

Jacobi Polynomials

orthogonalize the Beta Distribution

~~Pure Problem : Let $m_k =$~~

~~Applied Problem :~~

Moment Problems

Pure Math: Given a pdf $f(x)$ we define the moments $m_k = \int x^k f(x)\,dx$ $(m_0 = 1)$

Invert Problem: Given $m_0, m_1, \ldots$ is there ~~an~~ a pdf? Is it unique?

Exists is related to the infinite Hankel matrix $\begin{bmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & \\ m_2 & & \end{bmatrix}$ being pos definite

Uniqueness is implied by ~~Carl~~ Carleman's condition

Some variations & complications ...