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Group Representations + Zonal Polynomials

Consider the following as homog polynomials of deg k
in the elements of A: (B constant) (A, B sym)

		depends on eig(A) only
1)	$\operatorname{tr} A^k$	YES (obvious)
2)	$\operatorname{tr} (AB)^k$	NO in general
3)	$E_{Q \text{ Haar}} \operatorname{tr} (AQ^T B Q)^k$	YES (exact expression not obvious)

We often do not write explicit formulas
but we can:

1) $\operatorname{tr} A^k = \sum \lambda_i^k$

2) Take, for example, $B = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$

$\operatorname{tr} (AB)^k = A_{11}^k = \left(\sum_i \lambda_i q_i^2\right)^k,$

where q_i is the first component
of the eigenvectors of A

3) $k=1: \Rightarrow \frac{1}{n}(\operatorname{tr} A)(\operatorname{tr} B) = \frac{1}{n}\left(\sum \lambda_i\right) \operatorname{tr} B$

$k=2: \Rightarrow \left(\sum \lambda_i^2 - \frac{2}{n+1} \sum_{i<j} \lambda_i \lambda_j\right) * (\text{function of } B)$

<u>in general</u>:

$\operatorname{tr} (AQ^T B Q)^k = \operatorname{tr} ((QAQ^T) B)^k$

eigenvectors of A are integrated out

Observation:

The coefficients of the characteristic polynomial of A, $\det(tI - A)$,

can be written as polynomials in the elements of A or in the eigenvalues of A ($\pm$ elementary sym polynomials in λ_i)

Group Representations

Turning a non-singular matrix into a much bigger matrix — so big it is not usually written explicitly. Usually one checks that the operation is linear

Example:

Consider the vector space of homog deg k polynomials in x, y

$$p(x,y) = a_0 x^k + a_1 x^{k-1} y + a_2 x^{k-2} y^2 + \dots + a_{k-1} x y^{k-1} + a_k y^k$$

p can be identified with $\begin{pmatrix} a_0 \\ \vdots \\ a_k \end{pmatrix} \in \mathbb{R}^{k+1}$.

We can rotate the plane with

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

What is the polynomial in the new coordinate system? (Backwards)

$$P_{new} = a_0(cx+sy)^K + a_1(cx+sy)^{K-1}(-sx+cy) + \dots$$

$c = \cos\theta, \; s = \sin\theta$

If you multiply out one gets a homog poly of deg K in x, y

$$P_{new} = a_0' x^K + a_1' x^{K-1} y + \dots$$

One can write

$$\begin{pmatrix} a_0' \\ \vdots \\ a_K' \end{pmatrix} = \text{Matrix} \begin{pmatrix} a_0 \\ \vdots \\ a_K \end{pmatrix}$$

The Matrix has elements built from $\cos\theta, \sin\theta$, but it is a matrix.
To know it's a matrix one can check that

$$(c_1 P_1 + c_2 P_2)\left(Q\begin{pmatrix} x \\ y \end{pmatrix}\right) = c_1 P_1\left(Q\begin{pmatrix} x \\ y \end{pmatrix}\right) + c_2 P_2\left(Q\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

and never think about an element of the matrix.

Write the map from $P(x)$ to $P(Mx)$

as a linear map on polynomials as

T_M.

Thus $(T_m P)(x) = P(Mx)$

If M is a matrix, T_m is a linear function from polynomials of degree k to polynomials of degree k.

Clearly $T_{M_1 M_2} = T_{M_1} T_{M_2}$

& if M has an inverse

$T_{M^{-1}} = T_M^{-1}$.

Suppose we have the representation of the orthogonal group on even degree polynomials

$(T_Q P)(x) = P(Qx)$

Consider the subspace of polynomials

$$\sum \{ c\,(x^2+y^2)^{k/2} \}$$

Note that T_Q is invariant

on this space.

This defines a representation on the subspace.

Zonal Polynomials

Consider the polynomials in A,

$$\{C_\kappa(AB) : B \text{ invertible}\}$$

The vector space generated by this set may be denoted by V_κ. Most polynomials in V_κ depend on eigenvalues + eigenvectors though $C_\kappa(A)$ is in the set + is "zonal," i.e. only depends on eigenvalues.

For homog polynomials V_K in the elements of a matrix consider

$$T_X(\phi(A)) = \phi(X^{-1}A(X^{-1})^T)$$

the inverse just gives a good order.

then

$$T_{XY} = T_X T_Y$$

because composing and conjugating can be thought of composing the conjugation.

V_K breaks into irreducible subspaces

&

$$(\operatorname{tr} X)^K = \sum_{K \vdash K} C_K(X)$$

by projecting into the subspaces.