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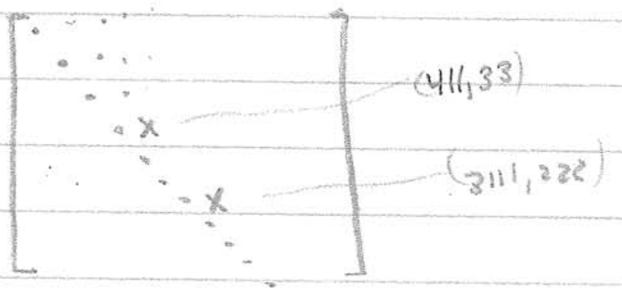
Jack Polynomials ($\alpha = 2$ Zonal Polynomials)

Many Definitions, None immediately Intuitive
Useful fact that Jacks are upper triangular in dominance ordering w.r.t. monomials:

$$J_\kappa = \sum_{\lambda \geq \kappa} U_{\kappa\lambda} M_\lambda$$

Partitions of 6: $6 - 51 - 42 < {411 \atop 33} > 321 < {3111 \atop 222} > 2211 - 21111 - 111111$

U =


Matrices that are upper triangular with a partial order form an algebra (closed under $+, -, *, \div$)

Probably a definition:

Let $P(X)$ be homog of degree k in eig X
The operator

$$\mathcal{L}P : P(A) \rightarrow \int_Q P(QAQ^T B)$$

creates a new polynomial that is homog of degree k in the eigenvalues of A with coefs that are polynomials in B.

Very probably this linear operator is triangular on monomials:

$$\int_Q m_\kappa(QAQ^TB) = \sum_{\lambda \geq \kappa} c_{\kappa\lambda}(B)\, m_\lambda(A)$$

The eigenvectors define the zonal polynomials. The eigenvalues are $\frac{J_\kappa(B)}{J_\kappa(I)}$

Why useful?

$$(\operatorname{tr} A)^k = \sum_{\kappa \vdash k} C_\kappa(A) \quad \text{"C normalization"}$$

Consider $A = \operatorname{randn}(m,n)$ $\quad W = A^TA$

$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$ $\quad \Sigma = D^2$

Joint Density of AD is

$$\left(\prod_{i,j} \frac{1}{\sqrt{2\pi d_j^2}}\right) e^{-\frac{1}{2}\sum a_{ij}^2/d_j^2}$$

$$= c\; e^{-\frac{1}{2}\operatorname{tr} W\Sigma^{-1}}$$

The Jack Polynomials become the Fourier Series for Random Matrix Theory

$$(dM) = c\,(U^T dU)^\wedge (V^T dV)^\wedge \prod |\sigma_i^2 - \sigma_j^2| \prod \sigma_i^{m-n}$$

$A = U \Sigma V^T$

$A^T A = V \Sigma^2 V^T$

$$c\, e^{-\frac{1}{2} \operatorname{Tr} V \Sigma^2 V^T D^{-2}} \qquad \prod |\sigma_i^2 - \sigma_j^2| \prod \sigma_i^{m-n} \quad (U^T dU)(V^T dV)$$

Clearly orthogonally invariant in U

What about V?

Need to integrate out

$$\sum \frac{1}{k!} \left(\frac{1}{2} \left(\operatorname{Tr} V \Sigma^2 V^T D^{-2} \right) \right)^k \quad (V^T dV)$$

Integrate term by term

$$\int_Q e^{\operatorname{tr} X Q Y Q^T} (Q^T dQ)^\wedge$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(X) C_\kappa(Y)}{k!\, C_\kappa(I)}$$

$${}_0F_0(X, Y)$$