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Intro to Jack Polynomials

- What they are
- Some Examples of what they are good for
- Many many applications waiting to be found

NOTATION

Partition Notation $\kappa \vdash 4$

$(4), (3\,1), (2\,2), (2\,1,1), (1,1,1,1)$

or $4^1, 1^1 3^1, 2^2, 1^2 2, 1^4$

Power Sums

$$P_4(x) = x_1^4 + x_2^4 + x_3^4 + \dots$$
$$P_4(X) = \operatorname{tr} X^4$$

$$P_k(x) = \sum x_i^k$$ ← Finite or Formally infinite sum

$$P_{2,1,1}(x) = (x_1^2 + x_2^2 + \dots)(x_1 + x_2 + \dots)^2$$
$$P_{2,1,1}(X) = \operatorname{tr} X^2 (\operatorname{tr} X)^2$$

$$P_\kappa(x) = \prod P_{\kappa_i}(x)$$
$$P_\kappa(X) = \prod \operatorname{Tr} X^{\kappa_i}$$

- Homogeneous Polynomials of deg K form a vector space V_K
- Power Sums are a basis for this space

Another basis is the Monomials

$$m_4(x) = \sum x_i^4 = p_4(x)$$

$$m_{2,1,1}(x) = \sum x_i^2 x_j x_k$$

i, j, k distinct
j < k

Many bases for scalar polynomials
x^k, Hermite, Laguerre, Jacobi
but only one homogeneous polynomial for each K
So exploring good bases is a true multivariate expedition

<u>Note</u>: Notions of $\kappa_1 < \kappa_2$ (Lexicographic, Dominance)

$$\begin{pmatrix} \text{vector} \\ \text{in} \\ p \\ \text{basis} \end{pmatrix} = \begin{pmatrix} L \end{pmatrix} \begin{pmatrix} \text{vector} \\ \text{in } m \\ \text{basis} \end{pmatrix}$$

respects dominance partial order(s)

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Ways to get to Sack Polynomials

Orthogonalize discrete inner product
"   integral "   "
eigenfunctions of dif operator
"   " integral operator
QR factorization of a scaled version of L
Maybe can pluck out of special integrals
~~Group Representation Theory~~ (R, C, H, ...)

So far: No one way has
been perfect for seeing what
they are good for, but they are
arising increasingly in combinatorics
Especially: Map Counting on Topological Surfaces

(4)

Stanley (McDonald)

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \alpha^{\ell(\lambda)}$$

$$z_\lambda = (1^{l_1} 2^{l_2} 3^{l_3} \dots) l_1! l_2! \dots$$

defines an inner product on V_K

$J^{(\alpha)}$ orthogonal &

$$\begin{pmatrix} \text{vector in} \\ J \\ \text{basis} \end{pmatrix} = \begin{pmatrix} \diagdown \; U \end{pmatrix} \begin{pmatrix} \text{vector in} \\ \text{monomial basis} \end{pmatrix}$$

respects
dominance
partial ordering

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$\alpha = 1$ for now ($\mathbb{R}$)

Muirhead:

Eigenfunctions of

$$\sum y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{i \neq j} \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i}$$

Irreducible Representations

of V_K

$$X \to L X L^T$$
$$\phi \to T(L)\phi$$

$$(T(L)\phi)X = \phi(L^{-1} X L^{-T})$$

$$(\mathrm{tr}\, X)^k = \sum_K C_K(X)$$

$$\int_{O(n)} J_K(X_1 Q X_2 Q^T)\,(Q^T dQ)^\wedge$$

$$= \frac{J_K(X_1)\, J_K(X_2)}{J_K(I)}$$

⑥

Should be holding X_2 constant

& thinking of

$$\int_{O(n)} f(X_1 \, Q \, X_2 \, Q^T)$$

as a linear function on polynomials in X_1

I think this is triangular

& zonals are

eigenfunctions.