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## Intro to Jack Polynomials

- What they are
- Some Examples of what they are good for
- Many many applications waiting to be found

Monomial Symmetric Polynomials

$$P_{\lambda} = \sum c_{\mu} x^{\mu}$$

### NOTATION

Partition Notation  $K=4$

(4), (3 1), (2 2), (2 1, 1), (1, 1, 1, 1)

or  $4^1, 1^3, 2^2, 1^2, 1^4$

### Power Sums

$$P_4(x) = x_1^4 + x_2^4 + x_3^4 + \dots$$

$$P_4(X) = \text{tr } X^4$$

$$P_K(x) = \sum x_i^K \quad \leftarrow \text{Finite or Formally infinite sum}$$

$$P_{2,1,1}(x) = (x_1^2 + x_2^2 + \dots)(x_1 + x_2 + \dots)^2$$

$$P_{2,1,1}(X) = \text{tr } X^2 (\text{tr } X)^2$$

$$P_{\lambda}(x) = \prod P_{K_i}(x)$$

$$P_{\lambda}(X) = \prod \text{tr } X^{K_i}$$

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- Homogeneous Polynomials of deg  $k$  form a vector space  $V_k$
- Powers Sums are a basis for this space

Another basis is the Monomials

$$M_k(x) = \sum x_i^k = P_k(x)$$

$$m_{2,1,1}(x) = \sum x_i^2 x_j x_k$$

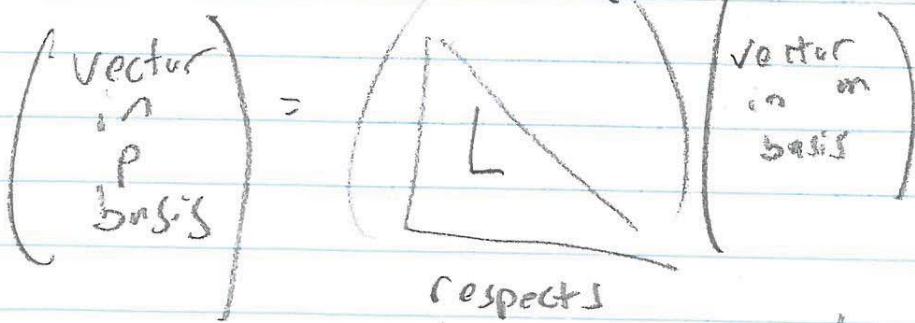
$\uparrow$   
 $k, j, k$  distinct  
 $j < k$

Many bases for scalar polynomials  
 $x^k$ , Hermite, Laguerre, Jacobi

but only one homogeneous polynomial  
for each  $k$

So exploring good bases is a true  
multivariate expedition

Note: Notions of  $\mathbb{N}_1 < \mathbb{N}_2$  (Lexicographic, Dominance)



respects  
dominance partial order

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Ways to get to Jack Polynomials

Orthogonalize discrete inner product  
" " integral " "

Eigenfunctions of dif operator

" " integral operator

QR Factorization of a scaled version of  $L$

Maybe can pluck out of special integrals

Group Representation Theory ( $\mathbb{R}, \mathbb{C}, \mathbb{H}, \dots$ )

So far: No one way has

been perfect for seeing what

they are good for, but they are

arising increasingly in combinatorics

Especially: Map counting on Topological Surfaces

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Stanley (McDonald)

$$\langle e_\lambda, e_\mu \rangle = \delta_{\lambda\mu} Z_\lambda \alpha^{e(\lambda)}$$

$$Z_\lambda = (1^{l_1} 2^{l_2} 3^{l_3} \dots) l_1! l_2! \dots \text{ENV}$$

defines an inner product on  $V_K$

$\mathcal{J}(\alpha)$  orthogonal  $\rightarrow$

$$\left( \begin{array}{c} \text{vector in} \\ \mathcal{J} \\ \text{basis} \end{array} \right) = \left( \begin{array}{c} \text{triangle} \\ \cup \end{array} \right) \left( \begin{array}{c} \text{vector on} \\ \text{monomial basis} \end{array} \right)$$

respects  
diagram  
partial ordering



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$d=1$  for now  $(\mathbb{R})$

Muirhead:

Eigenfunctions of

$$\sum y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{i < j} \frac{y_i^2 - y_j^2}{y_i - y_j} \frac{\partial}{\partial y_i}$$

Irreducible Representations

of  $V_K$

$$X \rightarrow LXL^T$$

$$\psi \rightarrow T(L)\psi$$

$$(T(L)\psi)(X) = \psi(L^{-1}XL^{-T})$$

$$\langle \psi, X \rangle^K = \sum_K c_K(X)$$

$$\int_{\mathcal{O}(n)} J_K(X_1, \psi X_2 \psi^T) (\psi^T \psi)^n$$

$$= \frac{J_K(X_1) J_K(X_2)}{J_K(I)}$$

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Should be holding  $x_2$  constant

+ thinking of

$$\int_{\mathcal{O}(n)} f(x_1, \Phi x_2, \Phi^T)$$

as a  $\underbrace{\text{polynomial}}$  linear function on  $x_1$

I think this is triangular

+ Zonals are

eigenfunctions.