Introduction to the Hankel-based model order reduction for linear systems

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State-space description

MIMO LTI CT dynamical system:

\[ \begin{align*}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*} \]

- \( x(t) \in \mathbb{R}^n \) – state
- \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{k \times n} \)
- \( u(t) : \mathbb{R} \to \mathbb{R}^m \) – inputs,
- \( y(t) : \mathbb{R} \to \mathbb{R}^k \) – outputs

Here we assume a system to be stable, i.e. matrix \( A \) is Hurwitz.
Model order reduction problem

Problem: find a dynamical system \( G^r(s) \) of a smaller degree \( q \) (McMillan degree – size of a minimal realization), such that the error of approximation \( e \) is “small” over all inputs!

Question: small in what sense??
Signals/system norms

$L_2$ - space of square-summable functions with 2-norm, or energy:

$$\| x(t) \|_2 = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sqrt{\int_{-\infty}^{\infty} x'(t)x(t) dt}$$

LTI system, as a linear operator on this space, has an induced 2-norm (maximum energy amplification, or $L_2$ gain), which equals $H$-infinity-norm of a system’s transfer function:

$$\| G(s) \|_{2-\text{induced}} \triangleq \sup_{u \in L_2} \frac{\| y(t) \|_2}{\| u(t) \|_2} = \sup_\omega \sigma_{\text{max}} (G(j\omega)) \triangleq \| G(s) \|_{\infty}$$

Now we know how to state our problem!
Model order reduction problem

Problem formulation:
find $G^r(s)$ of a smaller degree that minimizes

$$\| G(s) - G^r(s) \|_\infty$$

Unfortunately, we cannot solve this problem 😞
Instead, we use another system norm.
Hankel operator

- maps past inputs to future system outputs
- ignores any system response before time 0.
- Has finite rank (connection only by the state at \( t=0 \))
- As an operator on \( L_2 \) it has an induced norm (energy amplification)!
Hankel optimal MOR

Problem formulation:
find $G^r(s)$ of a smaller degree that minimizes

$$\left\| G(s) - G^r(s) \right\|_H$$

(Hankel norm of an error)

This problem has been solved and explicit algorithm is given for state-space LTI systems in Glover[84].
Controllability/observability

Hankel operator

P (controllability)
Which states are easier to reach?

Q (observability)
Which states produces more output?

LTI SYSTEM

Since we are interested in Hankel norm, we need to know how energy is transferred between input, state and output
Observability

How much energy in the output we shall observe if the system is released from some state \( x(0) \)?

\[
y(t) = Ce^{At} x(0)
\]

\[
\| y(t) \|_2^2 = x^T(0) \left( \int_0^\infty e^{A^T t} C^T C e^{A t} dt \right) x(0) = x^T(0)Qx(0)
\]

Observability Gramian satisfies Lyapunov equation:

\[
A^T Q + QA = -C^T C
\]

\( Q \) is SPD iff system is observable
What is the minimal energy of input signal needed to drive system to the state $x(0)$?

$$x(0) = \int_{-\infty}^{0} e^{-At} Bu(\tau)d\tau$$

$$\|u(t)\|_2^2 = x^T(0) \left( \int_{0}^{\infty} e^{A^T t} BB^T e^{A t} dt \right)^{-1} x(0) = x^T(0) P^{-1} x(0)$$

Controllability Gramian Satisfies Lyapunov equation:

$$AP + PA^T = -BB^T$$

$P$ is SPD iff system is controllable
Side note about Lyapunov equations

\[ AP + PA' = -R, \quad R \text{ - hermitian} \]

This equation has a unique solution \( P = P' \) if and only if:

\[ \forall i, j \quad \lambda_i(A) + \lambda_j(A) \neq 0 \]

Moreover, if \( R \) is SPD and \( A \) is Hurwitz, then \( P \) is SPD.

Assume some LTI CT system:

\[
\begin{align*}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

Lyapunov function:

\[ V(x) = x'Px > 0 \]

\[ \frac{dV(x(t))}{dt} = x^T (A^T P + PA)x < 0 \]

(reformulation of stability criterion)
Hankel singular values

- This operator has finite rank (equal to degree of $G$). Hankel singular values are square roots of an eigenvalues of the product $PQ$.
- If we approximate this operator by different one with lower rank, we cannot do better in the Hankel norm than the first removed HSV: $\| G(s) - G'(s) \|_H \geq \sigma_{q+1}(H_G)$.

Amazingly, this bound is tight (Adamjan et al., 71)
Truncated balanced reduction

Gramians are transformed with the change of basis as a quadratic forms:

\[ x \rightarrow Tx, \quad P \rightarrow TPT^T, \quad Q \rightarrow T^TQT^{-1} \]

We can find a basis, in which both gramians are equal and diagonal. Such transformation is called balancing transformation.

\[ P = R^TR \quad U\Sigma^2U^T = RQR^T \quad T = R^TU \Sigma^{-1/2} \]

\[ P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geq \ldots \geq \sigma_n \]
In the balanced realization we can perform truncation of the least observable and controllable modes!

\[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \]

\[ C = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \]

Truncated system \((A_{11}, B_1, C_1, D)\) will be stable (if \(\sigma_q \neq \sigma_{q+1}\), otherwise stable for almost all \(T\)) and have the following H-infinity error bound:

\[ \| G(s) - G^r(s) \|_\infty \leq 2 \sum_{k>q} \sigma_k(H) \]

"twice sum of a tail" rule
Truncated balanced reduction vs. Hankel optimal reduction

For the balanced truncation procedure we have the following error bounds:

\[ \| G(s) - G^r(s) \|_\infty \leq 2 \sum_{k>q} \sigma_k(H_G) \]
\[ \| G(s) - G^r(s) \|_H \leq 2 \sum_{k>q} \sigma_k(H_G) \]

TBR is not optimal in terms of the Hankel norm! For the Hankel optimal MOR the following bounds hold:

\[ \| G(s) - G^r(s) \|_\infty \leq \sum_{k>q} \sigma_k(H_G) \]
\[ \| G(s) - G^r(s) \|_H = \sigma_{q+1}(H_G) \]
References:

- Keith Glover, “All optimal Hankel-norm approximations of linear multivariable systems and their L-infinity error bounds”