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GLOBAL STABILIZATION OF PARTIALLY LINEAR COMPOSITE SYSTEMS*

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Abstract. A linear stabilizable, nonlinear asymptotically stable, cascade system is globally stabilizable by smooth dynamic state feedback if (a) the linear subsystem is right invertible and weakly minimum phase, and, (b) the only linear variables entering the nonlinear subsystem are the output and the zero dynamics corresponding to this output. Both of these conditions are coordinate-free and there is freedom of choice for the linear output variable. This result generalizes several earlier sufficient conditions for stabilizability. Moreover, the weak minimum-phase condition for the linear subsystem cannot be relaxed unless a growth restriction is imposed on the nonlinear subsystem.

Key words. composite systems, stabilization, Lyapunov function, nonlinear control

AMS(MOS) subject classifications. 93C10, 93C15, 93A20

1. Introduction. In this paper we propose new sufficient conditions for global stabilization, by means of state feedback, of *composite partially linear systems* in the form

$$\dot{x} = f(x, \xi), \qquad x \in R^n, \quad \xi \in R^q, \tag{1.1a}$$

$$\dot{\xi} = A\xi + Bu, \qquad u \in R^m, \tag{1.1b}$$

where $f(x, \xi)$ is a smooth (i.e., C^∞) function and A and B are constant matrices. Throughout the paper it is assumed that:

(H1) The pair (A, B) is stabilizable.

(H2) The equilibrium $x=0$ of $\dot{x}=f(x, 0)$ is globally asymptotically stable (GAS) and a smooth Lyapunov function $V(x)>0$, $x \neq 0$; $V(0)=0$, is known such that $V(x) \to \infty$ as $\|x\| \to \infty$

and

$$\nabla V(x) f(x, 0) < 0 \quad \text{for all } x \neq 0. \tag{1.2}$$

As a class of nonlinear composite systems [13], [19], [26], the partially linear systems (1.1) have become prominent because of recent results on partial feedback linearization, where $\dot{x}=f(x, 0)$ is referred to as the "nonlinear zero dynamics" [2]-[4], [9], [12]. It would appear that when $x=0$ is globally asymptotically stable as assumed by (H2), then the global stabilization of the whole system should not be difficult. Simple examples show that this is not so. Disturbed by an exponentially decaying input $\xi(t)$, the nonlinear system (1.1a) can become unstable, or even worse: its state may escape to infinity in finite time! One way to circumvent this difficulty is to restrict $f(x, \xi)$ by a global linear growth condition and then to apply the classical "total stability" theorems [7]. A criticism of the global linear growth assumption is that it does not let nonlinear systems be "nonlinear enough." It excludes simple chemical kinetics, mechanical phenomena such as Coriolis forces, etc.

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This paper continues the efforts of several recent studies [5], [10], [25], [22], [11], which do not make a linear growth assumption. Instead of constraining the nonlinear nature of $f(x, \xi)$, our approach characterizes its dependence on ξ by the expression

$$f(x, \xi)-f(x, 0)=G(x, \xi)C\xi. \tag{1.3}$$

Since for a given $f(x, \xi)$ the choice of G and C is not unique, we seek a smooth $n\times p$ matrix function $G(x, \xi)$ and a constant $p\times q$ matrix C to encompass the largest class of linear systems

$$\dot{\xi}=A\xi+Bu, \qquad \xi\in R^q, \quad u\in R^m, \tag{1.4a}$$

$$y=C\xi, \qquad y\in R^p. \tag{1.4b}$$

Our main result is a stable right invertibility (SRI) condition imposed on (1.4). This condition encompasses a much broader class of systems than the feedback positive real (FPR) condition of [11]. When the linear subsystem is not SRI, our second result imposes a restriction on the nonlinear subsystem, which is less severe than the linear growth condition.

The meaning of (H1) and (H2) is that each subsystem, taken isolated, is globally stable (or stabilizable). This setting is suitable for the construction of composite Lyapunov functions [13], [19], which we use to broaden the class of linear subsystems (1.4). In § 2 we start with a sum-composite Lyapunov function, leading to the class of stable invertible systems of relative degree one (SI_1). This class includes the FPR systems of [11], and is broadened by the assumption that the zero dynamics are stable ("weak minimum phase"), rather than asymptotically stable ("minimum phase"). The main result of § 3, and of the whole paper, removes the relative degree assumption and requires only that the linear subsystem (1.4) be stable right invertible. The analysis leading to this result provides new insights into linear system properties, revealed by the special coordinate basis (s.c.b.) of [17] and [23], which is our key analytical tool. As shown in § 4, the assumptions of the main theorem cannot be weakened unless some additional restrictions are imposed on $f(x, \xi)$. So, when the linear subsystem (1.4) is not SRI and the results of §§ 2 and 3 are not applicable, then § 4 introduces a constraint on the nonlinear subsystem.

2. The stabilization procedure in the case SI_1. The problem is to find a smooth feedback control

$$u=K\xi+v(x, \xi), \tag{2.1}$$

which guarantees the GAS property for the equilibrium $(x, \xi)=(0, 0)$ of the feedback system

$$\dot{x}=f(x, 0)+G(x, \xi)C\xi, \tag{2.2a}$$

$$\dot{\xi}=(A+BK)\xi+Bv(x, \xi). \tag{2.2b}$$

This system is obtained by applying the control (2.1) to the system (1.1) and taking into account the representation (1.2) of $f(x, \xi)$. The two subsystems clearly displayed in (2.2) are

$$\dot{x}=f(x, 0), \tag{2.3a}$$

$$\dot{\xi}=(A+BK)\xi \triangleq A_K\xi. \tag{2.3b}$$

By (H2) a Lyapunov function for the nonlinear subsystem (2.3a) is $V(x)$, while (H1) assures the existence of K such that $\operatorname{Re}\lambda(A_k)<0$. Hence, a Lyapunov function for the linear subsystem (2.3b) can be chosen as $\xi^T P\xi$, where $P=P^T>0$ is such that

$$PA_K + A_K^T P = -Q \leqq 0, \tag{2.4a}$$

$$(Q^{1/2}, A_K) \text{ detectable.} \tag{2.4b}$$

Our approach is to use $V(x)$ and $\xi^T P\xi$ to form a composite Lyapunov function $W(x,\xi)$ for the whole system (2.2). The simplest choice is

$$W(x,\xi) = V(x) + \xi^T P\xi. \tag{2.5}$$

Its derivative for (2.2) is

$$\dot W(x,\xi) = \nabla V(x)[f(x,0) + G(x,\xi)C\xi] - [\xi^T Q\xi - 2\xi^T PBv(x,\xi)]. \tag{2.6}$$

This expression is not informative because it contains the interconnection terms which are sign indefinite. However, if $G(x,\xi)$, C, P, and $v(x,\xi)$ can be found such that the interconnection terms in (2.6) are cancelled, then

$$\dot W(x,\xi) = \nabla V(x)f(x,0) - \xi^T Q\xi. \tag{2.7}$$

A sufficient condition for being able to achieve the cancellation is

$$B^T P = C. \tag{2.8}$$

Under this condition, the explicit form of v resulting in (2.7) is

$$v(x,\xi) = -\tfrac{1}{2}[\nabla V(x)G(x,\xi)]^T. \tag{2.9}$$

Remark 1. Assuming, without loss of generality, that B and C are of full rank, (2.8) implies the same number of inputs and outputs $p=m$. This restriction will be removed in § 3.

PROPOSITION 1. *Suppose there exists a K such that* (2.4) *and* (2.8) *are satisfied. Then the equilibrium* $(x,\xi)=(0,0)$ *of the system* (2.2) *with this K and* (2.9) *is* GAS.

Proof. It is clear from (2.7) that $\dot W(x,\xi)\leqq 0$ for all (x,ξ) and $\dot W(x,\xi)<0$ if $x\neq 0$. Moreover, $W(x,\xi)\geqq 0$ for all x and ξ and equality holds if and only if $(x,\xi)=(0,0)$. This establishes global stability of $(x,\xi)=(0,0)$, since $W(x,\xi)\to\infty$ as $\|(x,\xi)\|\to\infty$. To establish the GAS of the $(x,\xi)=(0,0)$ it suffices to show that, if $\gamma: t\to(x(t),\xi(t))$ is a complete trajectory of (2.2) along which $\dot W=0$, then it follows that $x(t)\equiv 0$ and $\xi(t)\equiv 0$. To begin with, $x(t)$ must be zero for all t, because $\dot W(x,\xi)<0$ unless $x=0$. Moreover, $x(t)\equiv 0$ implies that v defined by (2.9) vanishes along γ. Therefore, $t\to\xi(t)$ is a solution of $\dot\xi = A_K\xi$ and $\dot W(x,\xi) = -\xi^T(t)Q\xi(t)=0$ for all t. By the detectability assumption (2.4b) this implies $\xi(t)\equiv 0$ and, hence, $(x,\xi)=(0,0)$ is GAS. □

The above construction is a variant of the cancellation procedure used in the model reference adaptive control and goes back to [16] and [15].

With Proposition 1 the stabilization problem is reduced to that of the existence of a K satisfying (2.4) and (2.8). In [11] this issue was addressed indirectly, via a positive real property of (C, A_K, B). Here we will deal directly with the properties of the linear subsystem (1.4) induced by (2.4) and (2.8), such as invertibility, relative degree, and zero dynamics. Let us recall their definitions.

Invertibility. The linear system (1.4) is said to be invertible if, for any C^q function $y_{\text{ref}}(t)$, where q is an integer, there exist $u(t)$ and $\xi(0)$ such that $y(t)=y_{\text{ref}}(t)$ for all $t\in[0,\infty)$.

Relative degree. When (1.4) is "square," $p=m$, it is said to have scalar relative degree r if its first $r-1$ Markov parameters are zero, $CA^iB=0$ for $i=0,1,\cdots,r-2$, and $CA^{r-1}B$ is nonsingular. Equivalently, the system (1.4) has relative degree r if it is invertible and all of its infinite zeros are of order r.

Zero dynamics. Let V^* be the supremal (A,B)-invariant subspace in Ker C, and let R^* be the supremal (A,B)-controllability subspace in Ker C. The solutions $\xi(t)$ of (1.4) restricted for all $t\in[0,\infty]$ to V^*/R^* are called the zero dynamics of (1.4). When (1.4) is invertible its zero dynamics are equivalently defined as the solutions $\xi(t)$ satisfying $y(t)\equiv 0$ for all t.

Weak minimum phase. An invertible linear system (1.4) is said to be weak minimum phase, or, equivalently, stable invertible (SI), if its zero dynamics are stable in the sense of Lyapunov.

We are now in the position to completely characterize the class of linear systems (1.4) specified by (2.4) and (2.8).

PROPOSITION 2. *The following two statements are equivalent*:

(a) *For the system* (1.4) *there exists K satisfying* (2.4) *and* (2.8).

(b) *The system* (1.4) *is stabilizable, stable invertible and, moreover, its leading Markov parameter CB is symmetric positive definite.*

Proof. (a)$\rightarrow$(b). We postmultiply (2.8) by B and verify that $CB=B^TC^T>0$. Hence, (1.4) is invertible and has relative degree one. To prove the stable invertiblity (weak minimum phase) property of (1.4) we assume, without loss of generality, that (1.4) is in the special coordinate basis (s.c.b.)

$$\dot{\xi}_0=A_0\xi_0+A_1\xi_1, \tag{2.10a}$$

$$\dot{\xi}_1=D_0\xi_0+D_1\xi_1+CBu, \tag{2.10b}$$

$$y=\xi_1. \tag{2.10c}$$

This s.c.b. has evolved from early works [20], [14], and [6] and its general form is given in [17] and [23]. Noting that CB is nonsingular, the choice of u to achieve $\dot{\xi}_1=0$ for all t is obvious from (2.10b). With this choice, $\xi_1(0)=0$ implies $y(t)=\xi_1(t)\equiv 0$ for all $t\in[0,\infty)$, so that the zero dynamics of (2.10) are the solutions of

$$\dot{\xi}_0=A_0\xi_0. \tag{2.11}$$

Hence, the eigenvalues of A_0 are the invariant zeros of (2.10). A simple calculation reveals an important property induced by the cancellation condition (2.8). Under this condition, P for the system (2.10) is block diagonal, $P=\operatorname{diag}(P_0,P_1)$, where P_0 and P_1 are positive-definite matrices of dimensions $(q-m)\times(q-m)$ and $m\times m$, respectively. Because of this property and using any $K=(K_0,K_1)$ appropriately partitioned, the Q matrix in (2.4) is of the form

$$Q=-\begin{pmatrix} P_0A_0^T+A_0P_0 & * \\ * & * \end{pmatrix}. \tag{2.12}$$

By assumption (a) this matrix is positive semidefinite, which implies (see [1]) that

$$P_0A_0+A_0^TP_0\leqq 0. \tag{2.13}$$

Thus the zero dynamics are stable, which completes the proof of (a)$\rightarrow$(b).

(b)$\rightarrow$(a) Since the system is invertible and has relative degree one, we can represent it by (2.10). Moreover, the stable-invertibility assumption implies that the zero dynamics system (2.11) is stable. Without loss of generality we now let $A_0=\operatorname{diag}(A_{01},A_{02})$, where

$$\operatorname{Re}\lambda(A_{01})<0,\quad \operatorname{Re}\lambda(A_{02})=0,\quad \text{and}\quad A_{02}+A_{02}^T=0. \tag{2.14}$$

Then the system (2.10) is rewritten as

$$\dot{\xi}_{01} = A_{01}\xi_{01} + A_{11}\xi_1, \tag{2.15a}$$

$$\dot{\xi}_{02} = A_{02}\xi_{02} + A_{12}\xi_1, \tag{2.15b}$$

$$\dot{\xi}_1 = D_{01}\xi_{01} + D_{02}\xi_{02} + D_1\xi_1 + CBu, \tag{2.15c}$$

$$y = \xi_1. \tag{2.15d}$$

The Hurwitz property of A_{01} allows us to define $P_{01} = P_{01}^T > 0$ as the solution of

$$P_{01}A_{01} + A_{01}^T P_{01} = -I. \tag{2.16}$$

To prove the existence of K satisfying (2.4) and (2.8) we make a particular choice of $K = (K_{01}, K_{02}, K_1)$:

$$K = -[(CB)^{-1}D_{01} + A_{11}^T P_{01}, (CB)^{-1}D_{02} + A_{12}^T, (CB)^{-1}D_1 + \tfrac{1}{2}I]. \tag{2.17}$$

For this choice of K, the matrix A_K for (2.15) is

$$A_K = \begin{pmatrix} A_{01} & 0 & A_{11} \\ 0 & A_{02} & A_{12} \\ -(CB)A_{11}^T P_{01} & -(CB)A_{12}^T & -\tfrac{1}{2}CB \end{pmatrix}. \tag{2.18}$$

The substitution of this A_K and $P = \operatorname{diag}[P_{01}, I, (CB)^{-1}]$ into (2.4a) and (2.8) proves that they satisfy (2.4a) and (2.8) with $Q = \operatorname{diag}(I, 0, I)$. To prove that (2.4b) is also satisfied we use $Q^{1/2} = \left(\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & I \end{smallmatrix}\right)$ and test the observability of the pair $(Q^{1/2}, A_K)$. The stabilizability of (A, B), assumption (H1), implies the controllability of (A_{02}, A_{12}), and, hence, the matrix $[sI - A_{02}, A_{12}]$ is full rank for all complex s. It follows that

$$\operatorname{rank}\begin{bmatrix} Q^{1/2} \\ sI - A_K \end{bmatrix} = q \quad \text{for all complex } s. \tag{2.19}$$

Thus $(Q^{1/2}, A_K)$ is observable and (2.4b) is satisfied. □

Applying Propositions 1 and 2 to our stabilization problem we summarize the results of this section as follows.

THEOREM 1. *Suppose that for the composite system* (1.1), *with f represented by* (1.2), (H1) *and* (H2) *hold, and the linear subsystem* (1.4) *is invertible with relative degree one and weakly minimum phase* (SI_1). *Then there exists a feedback law such that the equilibrium* $(x, \xi) = (0, 0)$ *of the closed-loop system* (2.2) *is* GAS. *A particular form of this feedback law is* (2.1) *with* $v(x, \xi)$ *given by* (2.9) *and K given by* (2.17).

Proof. In the (SI_1) systems the matrix CB is nonsingular, while in Proposition 2 it is assumed that CB is symmetric positive definite. However, it follows from (2.10b) that, with a static precompensator $\bar{u} = (CB)^{-1}u$, both Propositions 1 and 2 are applicable to any (SI_1) system. Alternatively, the same effect can be achieved with the postcompensator $\bar{y} = (CB)^{-1}y$. □

A question raised by the following example is whether the weak minimum-phase condition required in Theorem 1 is in some sense necessary.

Example 1. For $c_2 > 0$ the linear subsystem in

$$\dot{x} = -x^3 - x^3 y, \quad \dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = u, \tag{2.20a}$$

$$y = c_1\xi_1 + c_2\xi_2 \tag{2.20b}$$

is invertible with relative degree one, and (H1), (H2) are satisfied. For $c_1 \geqq 0$ the weak minimum-phase assumption is satisfied and, by Theorem 1, the system (2.20) is globally asymptotically stabilizable. What if the weak minimum-phase condition of Theorem 1 is not satisfied, that is, $c_1 < 0$? Then, as shown in [11], for all initial states $(x_0, \xi_{10}, \xi_{20})$ such that

$$(1 - c_1\xi_{10})x_0^2 > \frac{c_1}{2c_2} \tag{2.21}$$

the system (2.19) fails to be asymptotically controllable to zero and therefore fails to be smoothly stabilizable.

We will return to this issue in § 4 and show that the weak minimum-phase condition is necessary in the sense that, in general, it cannot be weakened without a further restriction on $f(x, \xi)$.

3. The stabilization procedure in the case SRI. The first generalization of Theorem 1 and, at the same time, a step toward our main result, is a global stabilization condition for the system

$$\dot{x} = f(x, 0) + G(x, \xi_0, \xi_1)\xi_1, \qquad x \in R^n, \tag{3.1a}$$

$$\dot{\xi}_0 = A_0\xi_0 + A_1\xi_1, \qquad \xi_0 \in R^{q_0}, \tag{3.1b}$$

$$\dot{\xi}_1 = \xi_2, \qquad \xi_i \in R^m, \quad i = 1, \cdots, r, \tag{3.1c}$$

$$\vdots$$

$$\dot{\xi}_{r-1} = \xi_r, \qquad \xi \in R^q, \quad q = q_0 + rm, \tag{3.1d}$$

$$\dot{\xi}_r = u_r, \quad y = \xi_1, \quad u_r \in R^n, \quad y \in R^m. \tag{3.1e}$$

The linear part of this system is in the form to which every invertible relative degree r (SI_r) system (1.4) can be transformed using first the s.c.b. of [17], [23], and [24] and then a feedback transformation $u = (CA^{r-1}B)^{-1}(Fx + u_r)$, with an appropriate F. The zero dynamics of this linear system are defined by (3.1b) with $\xi_1 = 0$, and the weak minimum phase property (SI_r) implies that they are stable. To simplify notation, we assume that A_0 does not have an asymptotically stable part, i.e., we let

$$A_0^T + A_0 = 0. \tag{3.2}$$

There is no loss of generality here because if some of the linear zero dynamics are asymptotically stable, we simply incorporate them in the nonlinear subsystem (3.1a) with an obvious redefinition of x, f, and G. However, our next assumption, already satisfied by the special form of (3.1a), is essential.

(H3) In (1.1) the dependence of $f(x, \xi)$ on ξ is such that (1.3) has the form

$$f(x, \xi) - f(x, 0) = G(x, \xi_0, \xi_1)\xi_1, \tag{3.3}$$

that is, G is allowed to depend only on the output $y = \xi_1$ and the linear zero dynamics ξ_0 induced by this output.

This assumption is a structural characterization of the linear/nonlinear interconnection (1.3). A choice of $y = Cx = \xi_1$ uniquely specifies ξ_0 via its s.c.b. Then (3.3) may or may not be satisfied even when (1.3) is satisfied. Let us illustrate this point.

Example 2. For the system

$$\dot{x} = -x^3 - \xi_1(\alpha\xi_1 + \xi_2)x^3, \qquad \dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = u \tag{3.4}$$

the choice of G and C in (3.3) depends on α. If $\alpha \geqq 0$, then the choice $y = \alpha\xi_1 + \xi_2$ results in a linear stable invertible system with $r = 1$ so that Theorem 1 applies. If $\alpha < 0$

then the same linear system is nonminimum phase and Theorem 1 does not apply. So we must try the second choice $y=\xi_1$, resulting in a linear system with $r=2$ and trivially minimum phase, because it has no finite zeros. However, now the connection structure condition (H3) is not satisfied because $G=(\alpha\xi_1+\xi_2)x^3$ depends not only on ξ_1, but also on ξ_2. In Example 3 we will discuss an important implication of this violation of (H3).

Returning to (3.1) let us recall from Theorem 1 and (2.17) that for the case $r=1$ a stabilizing control for (3.1) with (3.2) is

$$u_1(x,\xi_0,\xi_1)=-A_1^T\xi_0-\tfrac{1}{2}\xi_1+v(x,\xi). \tag{3.5}$$

With these preliminaries out of the way, the stabilization condition or the case of relative degree r is obtained using the chain of integrators result [11], [10], [25], [22].

PROPOSITION 3. *Suppose that the composite system* (1) *satisfies* (H1) *and* (H2) *and that the linear subsystem* (1.4) *is invertible with relative degree r and weakly minimum phase* (SI_r). *If, in addition, the connection-structure condition* (H3) *is satisfied, then this composite system is globally asymptotically stabilizable at* $(x,\xi)=(0,0)$ *by a smooth state feedback control. Furthermore, the expressions for a stabilizing control and for a corresponding Lyapunov function can be derived recursively.*

Proof. It is sufficient to prove this statement for the system (3.1). Let us start with the case $r=2$. From the first three equations (3.1a)-(3.1c) the result would be known from Theorem 1, if ξ_2 were the control variable u_1 in (3.5). This suggests that ξ_2 be modified as follows:

$$\xi_2=u_1(x,\xi_0,\xi_1)+\tilde{\xi}_2,\qquad \tilde{\xi}^T=[\xi_0^T,\xi_1^T,\tilde{\xi}_2^T]. \tag{3.6}$$

The time derivative of u_1 along the solutions of (3.1) can be evaluated explicitly as a function of x and $\tilde{\xi}$. We denote it by

$$\left.\frac{du_1}{dt}\right|_{(3.1)}=h_1(x,\tilde{\xi}). \tag{3.7}$$

Then for $r=2$ the system (3.1) becomes

$$\dot{x}=f(x,0)+G(x,\xi_0,\xi_1)\xi_1, \tag{3.8a}$$

$$\dot{\xi}_0=A_0\xi_0+A_1\xi_1, \tag{3.8b}$$

$$\dot{\xi}_1=\tilde{\xi}_2+u_1(x,\xi_0,\xi_1), \tag{3.8c}$$

$$\dot{\tilde{\xi}}_2=-h_1(x,\xi_0,\xi_1)+u_2. \tag{3.8d}$$

For this system we use the Lyapunov function

$$W_2(x,\tilde{\xi})=V(x)+\|\tilde{\xi}\|^2. \tag{3.9}$$

Its time derivative for (3.8) is

$$\dot{W}=\nabla V(x)f(x,0)-\|\xi_1\|^2+2\tilde{\xi}_2^T[\xi_1-h_1(x,\tilde{\xi})+u_2]. \tag{3.10}$$

An obvious choice of u_2 that makes $\dot{W}\leqq 0$ is

$$u_2(x,\tilde{\xi})=-\xi_1-\tfrac{1}{2}\tilde{\xi}_2+h_1(x,\tilde{\xi}). \tag{3.11}$$

The remaining step of the proof that $(x,\xi_0,\xi_1,\tilde{\xi}_2)=(0,0,0,0)$ is the GAS equilibrium of (3.8) is, as in Proposition 2, via an observability property which is guaranteed by the controllability of (A_0,A_1). The return to the original coordinates via (3.6) shows that $\tilde{\xi}_2\to 0$ implies $\xi_2\to 0$, which completes the proof for $r=2$.

To proceed to the case $r=3$ we note that, if ξ_3 were the control variable, the result (3.11) for $r=2$ would apply, which in turn suggests the modification

$$\xi_3 = u_2(x, \xi_0, \xi_1, \xi_2) + \tilde{\xi}_3 \tag{3.12}$$

where u_2 is expressed using ξ_2 rather than $\tilde{\xi}_2$. Adding the term $\|\tilde{\xi}_3\|^2$ to W_2 the new Lyapunov function W_3 is formed. Requiring that $\dot{W}_3 \leqq 0$ we obtain a stabilizing $u_3(x, \tilde{\xi})$ for the case $r=3$. It is clear that this procedure can be continued for any r, which completes the proof. □

Once again, an example is used to illustrate the closeness of the sufficient condition above to being also necessary.

Example 3. Let us reexamine the system (3.4) in Example 2 in the case when $\alpha < 0$ and $y = \xi_1$. In this case $r=2$, but the connection structure (H3) is violated and Proposition 3 does not apply. A detailed calculation in [11] shows that in this case there are initial conditions $\{x(0), \xi_1(0), \xi_2(0)\}$ for which the solutions of (3.4) are either unbounded as $t \to \infty$ or escape to infinity in finite time. It follows that for the system (3.4) the assumption (H3) cannot be relaxed to allow G to depend on both ξ_1 and ξ_2.

We are now prepared to remove the assumption that the linear system is "square," that is, $m=p$, and with a scalar relative degree. In the next step we allow $m \geqq p$ and require that the linear subsystem be right invertible and weakly minimum phase. The definitions of right invertibility and weak minimum phase are the same as in § 2 except that now we have $m \geqq p$. The problem of converting a right invertible system into an invertible one with scalar relative degree, which has been examined during the last two decades (e.g., [27], [21]), involves dynamic decoupling via precompensator and static feedback. In the following proposition, this conversion is achieved with the preservation of the weak minimum phase property using the results of [23] and [24].

PROPOSITION 4. *Consider the system* (1.4) *with* $m \geqq p$. *Assume that* (1.4) *is right invertible and let* $H(s)$ *be its transfer function matrix. Then there exists a precompensator* $u = C(s)\bar{u}$, $\bar{u} \in R^p$, *such that the system* $\bar{H}(s) \triangleq H(s)C(s)$ *has the following properties*:

(i) $\bar{H}(s)$ *has relative degree* r.

(ii) *Invariant zeros of* $\bar{H}(s)$ = *invariant zeros of* $H(s) \cup \Lambda$,

where Λ *denotes the set of additional invariant zeros induced by the compensator* $C(s)$ *and arbitrarily assignable.*

Proof. In the proof we construct two precompensators. The task of the first precompensator $u = C_1(s)\hat{u}$ is to "square down" $\hat{H}(s) \triangleq H(s)C_1(s)$ subject to the requirement that the "squared" system satisfies (ii). The task of the second precompensator $C_2(s)$ is that the compensated system $\bar{H}(s)$ be of relative degree r, but without changing the finite-zero structure of $\hat{H}(s)$. In other words, we require that invariant zeros of $\bar{H}(s)$ equal invariant zeros of $\hat{H}(s)$.

As the design of $C_1(s)$ was developed in [24], the remaining task is to design $C_2(s)$. Since $\hat{H}(s)$ is invertible, it can be represented in the s.c.b. of [23] as follows:

$$\dot{\xi}_0 = A_0\xi_0 + A_1\tilde{y}, \tag{3.12a}$$

$$\dot{\xi}_i = A_i\xi_i + B_i\left(\tilde{u}_i + \sum_{j=0}^{r} D_{ij}\xi_j\right) + L_i\tilde{y}, \qquad i = 1, \cdots, r, \tag{3.12b}$$

$$\tilde{y} = C_i\xi_i, \quad \tilde{y}_i \in R^{q_i}, \quad \tilde{y}^T = (\tilde{y}_1^T, \cdots, \tilde{y}_r^T), \quad \tilde{y} = \Gamma_1 y, \tag{3.12c}$$

$$\tilde{u}^T = (\tilde{u}_1^T, \cdots, \tilde{u}_r^T), \qquad \hat{u} = \Gamma_2\tilde{u}, \tag{3.12d}$$

where $\Gamma_1, \Gamma_2 \in R^{m \times m}$ are nonsingular matrices and

$$A_i = \begin{pmatrix} 0 & I_{(i-1)q_i} \\ 0 & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ I_{q_i} \end{pmatrix}, \quad C_i = (I_{q_i}, 0). \tag{3.13}$$

This s.c.b. displays the zero structure of the system:

Invariant zeros of $\hat{H}(s)$ = eigenvalues of A_0,
Zero dynamics of $\hat{H}(s)$ = the solutions of $\dot{\xi}_0 = A_0\xi_0$,
i = order of an infinite zero, iq_i = number of infinite zeros of order i.

Now, to design $C_2(s)$ that makes $\hat{H}(s)$ of relative degree r we simply add an appropriate number of integrators to each input $\tilde{u}_i$. Hence we let

$$\bar{u}^T = (\bar{u}_1^T, \cdots, \bar{u}_r^T), \quad \tilde{u}_i = \frac{1}{s^{r-i}}\bar{u}_i, \quad \tilde{u} = \tilde{C}_2(s)\bar{u}, \quad \tilde{C}_2(s) \triangleq \operatorname{diag}\left(\frac{1}{s^{r-i}}\right) \tag{3.14}$$

and obtain that $\hat{H}(s)\Gamma_2\tilde{C}_2(s)$ has relative degree r and its invariant zeros are the invariant zeros of $\hat{H}(s)$. So the second compensator is $C_2(s) \triangleq \Gamma_2\tilde{C}_2(s)$. □

Applying Propositions 3 and 4 to our stabilization problem we formulate the main result of this paper as follows.

THEOREM 2. *If the assumptions* (H1)–(H3) *hold, and the linear subsystem* (1.4) *is right invertible and weakly minimum phase, then the composite system* (1.1) *is globally asymptotically stabilizable at* $(x, \xi) = (0, 0)$ *by dynamic state feedback.*

4. Restrictions on the nonlinear part. An assumption made throughout this paper is that the full state of the composite system (1.1) is available for feedback. Despite this assumption, our stabilizability conditions impose restrictions on the input-output structure of the linear subsystem. In addition to the connection structure and right invertibility assumptions, the key restriction is that the linear subsystem be *weakly minimum phase.* The analysis of Example 1 has given us a hint that this key restriction is in some sense necessary. Pursuing this hint we now prove that, given a strictly nonminimum phase linear subsystem (1.4), a nonlinear subsystem can be found such that the cascade (1.1) of these two subsystems, satisfying (H1)–(H3), is not globally stabilizable. Our Theorem 3 reveals that the underlying instability mechanism is an interplay of unstable zero dynamics with rapidly growing nonlinear terms, such as x^3. To limit this interplay, in Proposition 5 we introduce a specific growth condition which is less restrictive than a global Lipshitz condition.

THEOREM 3. *Consider the composite system satisfying assumption* (H1)–(H3):

$$\dot{x} = f(x, 0) + G(x, \xi_0, y)y, \qquad x \in R^n, \tag{4.1a}$$

$$\dot{\xi} = A\xi + Bu, \qquad \xi \in R^q, \quad u \in R^m, \tag{4.1b}$$

$$y \in C\xi, \qquad y \in R^p, \tag{4.1c}$$

and let the dynamics of (4.1b), (4.1c) *associated with its invariant zeros be represented by*

$$\dot{\xi}_0 = A_0\xi_0 + A_1 y, \qquad \xi_0 \in R^q. \tag{4.2}$$

When (4.1b), (4.1c) *is strictly nonminimum phase, i.e., some of the eigenvalues of* A_0 *have positive real parts, then there exist* $f(x, 0)$ *and* $G(x, \xi_0, y)$ *satisfying* (H2) *and* (H3) *such that the composite system* (4.1) *is not globally stabilizable.*

Proof. Without loss of generality we assume that all the eigenvalues of A_0 are with positive real parts $\operatorname{Re}\lambda(A_0) > 0$. (If only some of them are, then we let $A_0 = \operatorname{diag}(A_{01}, A_{02})$, with $\operatorname{Re}\lambda(A_{02}) > 0$ and modify the proof to apply to the subsystems with A_{02} instead of with A_0.) Using the positive-definite P_0 satisfying

$$P_0A_0 + A_0^TP_0 = 2I, \tag{4.3}$$

we evaluate the derivative of $V_0=\xi_0^T P_0 \xi_0$ along the trajectories of (4.2):

$$\begin{aligned}\dot V_0 &= 2\|\xi_0\|^2+2\xi_0^T P_0 A_1 y+\|P_0A_1y\|^2-\|P_0A_1y\|^2\\ &\geqq \|\xi_0\|^2-\|P_0A_1y\|^2\\ &\geqq \beta_1 V_0-\beta_2\|A_1y\|^2\end{aligned} \tag{4.4}$$

where $\beta_1=1/\lambda_{\max}(P_0)$, $\beta_2=\|P_0\|^2$. We are now in the position to pick a nonlinear subsystem to complete the proof. Consider the nonlinear subsystem defined by

$$f(x,0)=-x^3, \qquad G(x,\xi_0,y)=x^3\beta_2 y^T A_1^T A_1, \tag{4.5}$$

which satisfies (H2), (H3) and is nontrivial because, in view of (H1), the pair (A_0, A_1) is completely controllable and, hence, $A_1\neq 0$. Integrating the nonlinear subsystem

$$\dot x=-x^3+\beta_2 y\|A_1y\|^2x^3 \tag{4.6}$$

we obtain

$$2x^2(t)=\left[\frac{1}{2x^2(0)}+t-\int_0^t \beta_2\|A_1y(s)\|^2\,ds\right]^{-1}\triangleq\frac{1}{\theta(t)}. \tag{4.7}$$

Clearly, $\theta(0)>0$ and $\theta(t)$ must remain nonnegative for all $t>0$ or else $x(t)$ would escape to infinity. Thus, using (4.4) a necesary condition for $x(t)$ to remain bounded is

$$\frac{1}{2x^2(0)}+t+\int_0^t (\dot V_0(s)-\beta_1 V_0(s))\,ds\geqq 0 \tag{4.8}$$

and, hence,

$$V_0(t)\geqq \beta_1\int_0^t\left(V_0(s)-\frac{1}{\beta_1}\right)ds+V_0(0)-\frac{1}{2x^2(0)}. \tag{4.9}$$

Finally, applying a version of Gronwall's lemma, (4.9) implies that

$$V_0(t)\geqq\frac{1}{\beta_1}+\left(V_0(0)-\frac{1}{2x^2(0)}-\frac{1}{\beta_1}\right)e^{\beta_1 t}. \tag{4.10}$$

Now, from $V_0(0)=\xi^T(0)P_0\xi(0)$ we observe that, for any given $x(0)$, there exists $\xi(0)$ such that the factor multiplying $e^{\beta_1 t}$ is positive and $V_0(t)=\xi_0^T(t)P_0\xi_0(t)$ grows exponentially. This completes the proof, because $\theta(t)\geqq 0$, a necessary condition for boundedness of $x(t)$, implies that $\xi_0(t)$ grows unbounded as $t\to\infty$. For $\xi_0(t)$ to remain bounded, $\theta(t)$ must become negative at some finite time at which $x(t)$ escapes to infinity. □

While Theorem 3 shows the limits to stabilizability of the composite system caused by the nonminimum phase property of its linear part, the above proof reveals the underlying instability mechanism. The effort to stabilize the unstable linear zero dynamics may destabilize the nonlinear subsystem through some rapidly growing nonlinear connection terms. It is clear, therefore, that the class of nonlinear subsystems which can be cascaded with linear nonminimum phase subsystem must be restricted by restricting the growth of the connection terms. It turns out that, under one such restriction, the feedback loop needs to be closed only around the linear subsystem. With $u=K\xi$ and $v(x,\xi)=0$, the feedback system (2.2) becomes

$$\dot x=f(x,0)+G(x,\xi)\xi=f(x,\xi), \tag{4.11a}$$

$$\dot\xi=(A+BK)\xi=A_K\xi, \tag{4.11b}$$

where the decomposition of $f(x, \xi)$ in (4.11a) is always possible due to the smoothness of $f(x, \xi)$. The assumption (H2) is now strengthened by requiring global exponential stability (GES), rather than only global asymptotic stability of $\dot{x}=f(x, 0)$. Another crucial restriction to be imposed is the following:

(H4) There exists a nondecreasing scalar function $\gamma(\|\xi\|)\geqq 0$, bounded for all bounded ξ, such that

$$\|G(x, \xi)\| \leqq \gamma(\|\xi\|)\|x\| \quad \text{for all } x, \xi. \tag{4.12}$$

This assumption is much less restrictive than the linear growth condition of [18]. It includes, for example, the product nonlinearities such as $G(x, \xi)\xi = \xi^2 x$.

PROPOSITION 5. *If $x=0$ is the* GES *equilibrium of $\dot{x}=f(x, 0)$ and* (H1) *and* (H4) *hold, then the equilibrium $(x, \xi)=(0, 0)$ of the composite feedback system* (4.1) *is* GES *for every linear feedback $u = K\xi$ such that* $\operatorname{Re} \lambda(A_K)<0$.

Proof. In view of the GES assumption, the Lyapunov function $V(x)$ defined in (H2) has the following additional properties:

$$\alpha_1\|x\|^2 \leqq V(x) \leqq \alpha_2\|x\|^2, \qquad \|\nabla V(x)\| \leqq \alpha_3\|x\|, \tag{4.13}$$

$$\dot{V} \leqq -\alpha_0 V, \tag{4.14}$$

where $\dot{V}$ is the derivative of V for $\dot{x}=f(x, 0)$ and $\alpha_0, \cdots, \alpha_3$ are some positive constants. Taking the derivative of V for (4.11b) we obtain

$$\dot{V}(x, t)=\nabla V(x) f(x, 0)+\nabla V(x) G(x, \xi(t))\xi(t), \tag{4.15}$$

where any solution $\xi(t)$ of (4.11b) satisfies

$$\|\xi(t)\| \leqq k\, e^{-at}\|\xi(0)\|, \qquad k \geqq 1, \quad a>0. \tag{4.16}$$

Taking into account (4.12), (4.13), (4.14), and (4.16) we obtain from (4.15)

$$\dot{V} \leqq -\alpha_0 V+\frac{\alpha_3 \gamma(k\|\xi(0)\|)}{\alpha_1}\|\xi(0)\|\, e^{-at} V. \tag{4.17}$$

From this inequality it follows that $V(x(t))$ is bounded by

$$V(x(t)) \leqq k_0(\xi(0))\, e^{-\alpha_0 t} V(x(0)), \tag{4.18}$$

where $k_0(\xi(0))=\exp\{(\alpha_3 k/\alpha_0 a)\gamma(k\|\xi(0)\|)\|\xi(0)\|\}$. This completes the proof of global exponential stability of (4.11). □

5. Conclusions. The two types of structure constraints imposed by the coordinate-free stability condition of Theorem 2 are, first, the interconnection structure constraint and, second, the linear stable right invertibility constraint. To examine the first constraint, consider a decomposition $f(x, \zeta)=f_0(x, \xi)+R(x, \xi)$ that is more general than (3.3). A simple extension of the assumption (H2) is to require for $\dot{x}=f_0(x, \xi)$ that the asymptotic stability property, guaranteed by $V(x)$, be uniform in ξ. Much more fundamental is the question of whether an assumption about the interconnection $R(x, \xi)$, less restrictive than (H3), can be made. Once a linear subsystem output $y=C\xi=\xi_1$ is chosen, the assumption (H3) disallows $R(x, \xi)$ to depend on linear variables other than ξ_1 and the zero dynamics ξ_0 induced by the output ξ_1. For linear systems with relative degree two and higher, this restriction is a challenging research topic. If, as our Example 3 suggests, the interconnection condition (H3) is in some cases necessary, then the challenge is to delineate such cases, and to search for less restrictive conditions for other classes of systems. In any event, the study of delicate interconnection properties, initiated in [11] and in this paper, is a promising direction for future research.

The second condition, which restricts the linear subsystem to be right invertible and weakly minimum phase, cannot be relaxed, without imposing some form of growth restriction on the nonlinear subsystem, as shown in Theorem 3 and Proposition 5. A direction in which the right invertibility condition can be generalized is to consider that both subsystems in the cascade are nonlinear and the first one is right invertible and globally minimum phase. The results of this paper combined with several nonlinear invertibility results starting with [8], justify the conjecture that a nonlinear analogue of Theorem 2 exists, at least for the minimum phase case.

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