Projection Operator in Adaptive Systems

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Abstract

The projection algorithm is frequently used in adaptive control and this note presents a detailed analysis of its properties.

1 Introduction

These notes started in [2] as a personal communication from Eugene to colleagues in the field of adaptive control and summarized results from [5, 3, 1, 4]. Properties of the projection operator are explored in detail in the following section.

2 Properties of Convex Sets and Functions

Definition 1. A set $E \subset \mathbb{R}^k$ is convex if

$$\lambda x + (1 - \lambda)y \in E$$

whenever $x \in E$, $y \in E$, and $0 \leq \lambda \leq 1$

Remark. Essentially, a convex set has the following property. For any two points $x, y \in E$ where $E$ is convex, all the points on the connecting line from $x$ to $y$ are also in $E$.

Definition 2. A function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

$\forall 0 \leq \lambda \leq 1$.

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Lemma 3. Let $f(\theta) : \mathbb{R}^k \to \mathbb{R}$ be a convex function. Then for any constant $\delta > 0$ the subset $\Omega_\delta = \{\theta \in \mathbb{R}^k | f(\theta) \leq \delta\}$ is convex.

Proof. Let $\theta_1, \theta_2 \in \Omega_\delta$. Then $f(\theta_1) \leq \delta$ and $f(\theta_2) \leq \delta$. Since $f(x)$ is convex then for any $0 \leq \lambda \leq 1$

$$f(\lambda \theta_1 + (1-\lambda) \theta_2) \leq \lambda f(\theta_1) + (1-\lambda) f(\theta_2) \leq \lambda \delta + (1-\lambda) \delta = \delta$$

\therefore f(\theta) \leq \delta, thus, $\theta \in \Omega_\delta$.

Lemma 4. Let $f(\theta) : \mathbb{R}^k \to \mathbb{R}$ be a continuously differentiable convex function. Choose a constant $\delta > 0$ and consider $\Omega_\delta = \{\theta \in \mathbb{R}^k | f(\theta) \leq \delta\} \subset \mathbb{R}$. Let $\theta^*$ be an interior point of $\Omega_\delta$, i.e. $f(\theta^*) < \delta$. Choose $\theta_b$ as a boundary point so that $f(\theta_b) = \delta$. Then the following holds:

$$(\theta^* - \theta_b)^T \nabla f(\theta_b) \leq 0 \tag{1}$$

where $\nabla f(\theta_b) = \left(\frac{\partial f(\theta)}{\partial \theta_1} \ldots \frac{\partial f(\theta)}{\partial \theta_k}\right)^T$ evaluated at $\theta_b$.

Proof. $f(\theta)$ is convex \therefore

$$f(\lambda \theta^* + (1-\lambda) \theta_b) \leq \lambda f(\theta^*) + (1-\lambda) f(\theta_b)$$

equivalently,

$$f(\theta_b + \lambda (\theta^* - \theta_b)) \leq f(\theta_b) + \lambda (f(\theta^*) - f(\theta_b))$$

For any $0 < \lambda \leq 1$:

$$\frac{f(\theta_b + \lambda (\theta^* - \theta_b)) - f(\theta_b)}{\lambda} \leq f(\theta^*) - f(\theta_b) \leq \delta - \delta = 0$$

and taking the limit as $\lambda \to 0$ yields (1).

3 Projection

Definition 5. The Projection Operator for two vectors $\theta, y \in \mathbb{R}^k$ is now introduced as

$$\text{Proj}(\theta, y, f) = \begin{cases} y - \frac{(y - \nabla f(\theta)(\nabla f(\theta))^T)}{\|\nabla f(\theta)\|^2} y f(\theta) & \text{if } f(\theta) > 0 \land y^T \nabla f(\theta) > 0 \\ y & \text{otherwise.} \end{cases} \tag{2}$$

where $f : \mathbb{R}^k \to \mathbb{R}$ is a convex function and $\nabla f(\theta) = \left(\frac{\partial f(\theta)}{\partial \theta_1} \ldots \frac{\partial f(\theta)}{\partial \theta_k}\right)^T$. Note that the following are notationally equivalent $\text{Proj}(\theta, y) = \text{Proj}(\theta, y, f)$ when the exact structure of the convex function $f$ is of no importance.
Remark. A geometrical interpretation of (2) follows. Define a convex set $\Omega_0$ as

$$\Omega_0 \triangleq \{ \theta \in \mathbb{R}^k | f(\theta) \leq 0 \}$$  \hspace{1cm} (3)

and let $\Omega_1$ represent another convex set such that

$$\Omega_1 \triangleq \{ \theta \in \mathbb{R}^k | f(\theta) \leq 1 \}$$  \hspace{1cm} (4)

From (3) and (4) $\Omega_0 \subset \Omega_1$. From the definition of the projection operator in (7) $\theta$ is not modified when $\theta \in \Omega_0$. Let

$$\Omega_A \triangleq \Omega_1 \setminus \Omega_0 = \{ \theta | 0 < f(\theta) \leq 1 \}$$

represent an annulus region. Within $\Omega_A$ the projection algorithm subtracts a scaled component of $y$ that is normal to boundary $\{ \theta | f(\theta) = \lambda \}$. When $\lambda = 0$, the scaled normal component is 0, and when $\lambda = 1$, the component of $y$ that is normal to the boundary $\Omega_1$ is entirely subtracted from $y$, so that $\text{Proj}(\theta, y, f)$ is tangent to the boundary $\{ \theta | f(\theta) = 1 \}$. This discussion is visualized in Figure 1.

![Figure 1: Visualization of Projection Operator in $\mathbb{R}^2$.](image)

Remark. Note that $(\nabla f(\theta))^T \text{Proj}(\theta, y, f) = 0$ whenever $f(\theta) = 1$ and that the general structure of the algorithm is as follows

$$\text{Proj}(\theta, y) = y - \alpha(t)\nabla f(\theta)$$ \hspace{1cm} (5)

for some time varying $\alpha$ when the modification is triggered. Multiplying the left hand side of the equation by $(\nabla f(\theta))^T$ and solving for $\alpha$ one finds that

$$\alpha(t) = ((\nabla f(\theta))^T \nabla f(\theta))^{-1} (\nabla f(\theta))^T y$$ \hspace{1cm} (6)

and thus the algorithm takes the form

$$\text{Proj}(\theta, y) = y - \nabla f(\theta) ((\nabla f(\theta))^T \nabla f(\theta))^{-1} (\nabla f(\theta))^T y f(\theta)$$ \hspace{1cm} (7)

where the modification is active. Notice that the $f(\theta)$ has been added to the definition, making (7) continuous.

Lemma 6. One important property of the projection operator follows. Given $\theta^* \in \Omega_0,

$$(\theta - \theta^*)^T (\text{Proj}(\theta, y, f) - y) \leq 0.$$ \hspace{1cm} (8)
Proof. Note that

\[(\theta - \theta^*)(\text{Proj}(\theta, y, f) - y) = (\theta^* - \theta)^T(y - \text{Proj}(\theta, y, f))\]

If \(f(\theta) > 0 \land y^T \nabla f(\theta) > 0\), then

\[(\theta^* - \theta)^T\left(y - \left(y - \frac{\nabla f(\theta)(\nabla f(\theta))^T}{\|\nabla f(\theta)\|^2} y f(\theta)\right)\right)\]

and using Lemma 4

\[
\frac{(\theta^* - \theta)^T \nabla f(\theta)(\nabla f(\theta))^Ty}{\|\nabla f(\theta)\|^2} \geq 0 \quad \text{if } f(\theta) \leq 0
\]

otherwise \(\text{Proj}(\theta, y, f) = y\). \(\square\)

**Definition 7 (Projection Operator).** The general form of the projection operator is the \(n \times m\) matrix extension to the vector definition above.

\[
\text{Proj}(\Theta, Y, F) = [\text{Proj}(\theta_1, y_1, f_1) \ldots \text{Proj}(\theta_m, y_m, f_m)]
\]

where \(\Theta = [\theta_1 \ldots \theta_m] \in \mathbb{R}^{n \times m}, Y = [y_1 \ldots y_m] \in \mathbb{R}^{n \times m}, \text{ and } F = [f_1(\theta_1) \ldots f_m(\theta_m)]^T \in \mathbb{R}^{m \times 1}\). Recalling (2)

\[
\text{Proj}(\theta_j, y_j, f_j) = \begin{cases} 
    y_j - \frac{\nabla f_j(\theta_j)(\nabla f_j(\theta_j))^T}{\|\nabla f_j(\theta_j)\|^2} y_j f_j(\theta_j) & \text{if } f_j(\theta_j) > 0 \land y_j^T \nabla f_j(\theta_j) > 0 \\
    y_j & \text{otherwise}
\end{cases}
\]

\(j = 1 \to m\).

**Lemma 8.** Let \(F = [f_1 \ldots f_m]^T \in \mathbb{R}^{m \times 1}\) be a convex vector function and \(\hat{\Theta} = [\hat{\theta}_1 \ldots \hat{\theta}_m], \Theta = [\theta_1 \ldots \theta_m], Y = [y_1 \ldots y_m]\) where \(\hat{\Theta}, \Theta, Y \in \mathbb{R}^{n \times m}\) then,

\[
\text{trace} \left\{ (\hat{\Theta} - \Theta)^T \left(\text{Proj}(\hat{\Theta}, Y, F) - Y\right) \right\} \leq 0.
\]

Proof. Using (8),

\[
\text{trace} \left\{ (\hat{\Theta} - \Theta)^T \left(\text{Proj}(\hat{\Theta}, Y, F) - Y\right) \right\} = \sum_{j=1}^{m} (\hat{\theta}_j - \theta_j)^T\left(\text{Proj}(\hat{\theta}_j, y_j, f_j) - y_j\right) \leq 0 \quad \square
\]

The application of the projection algorithm in adaptive control is explored below.

**Lemma 9.** If an initial value problem, i.e. adaptive control algorithm with adaptive law and initial conditions, is defined by:

1. \(\dot{\theta} = \text{Proj}(\theta, y, f)\)
2. \(\theta(t = 0) = \theta_0 \in \Omega_1 = \{\theta \in \mathbb{R}^k | f(\theta) \leq 1\}\)
Thus therefore

Substitution of (9) into (2) leads to

\[ \dot{f}(\theta) = (\nabla f(\theta))^T \text{Proj}(\theta, y, f) \]

(9)

Substitution of (9) into (2) leads to

\[ \dot{f}(\theta) = (\nabla f(\theta))^T \text{Proj}(\theta, y, f) \]

\[ = \begin{cases} 
(\nabla f(\theta))^T y(1 - f(\theta)) & \text{if } f(\theta) > 0 \land y^T \nabla f(\theta) > 0 \\
(\nabla f(\theta))^T y & \text{if } f(\theta) \leq 0 \lor y^T \nabla f(\theta) \leq 0
\end{cases} \]

therefore

\[ \begin{cases} 
\dot{f}(\theta) > 0 & \text{if } 0 < f(\theta) < 1 \land y^T \nabla f(\theta) > 0 \\
\dot{f}(\theta) = 0 & \text{if } f(\theta) = 1 \land y^T \nabla f(\theta) > 0 \\
\dot{f}(\theta) < 0 & \text{if } f(\theta) \leq 0 \lor y^T \nabla f(\theta) \leq 0
\end{cases} \]

Thus \( f(\theta_0) \leq 1 \Rightarrow f(\theta) \leq 1 \forall t \geq 0. \)

**Remark.** Given \( \theta_0 \in \Omega_0, \) \( \theta \) may increase up to the boundary where \( f(\theta) = 1. \) However, \( \theta \) never leaves the convex set \( \Omega_1. \)

**Example 10** (Projection Algorithm in Adaptive Control Law). Let \( \Theta(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^{m \times n} \) represent a time varying feedback gain in a dynamical system. This feedback gain is implemented as:

\[ u = \Theta(t)^T x \]

where \( u \in \mathbb{R}^n \) represents the control input and \( x \in \mathbb{R}^m \) the state vector. The time varying feedback gain is adjusted using the following adaptive law

\[ \dot{\Theta} = \text{Proj}(\Theta, -xe^T PB, F) \]

where \( e \in \mathbb{R}^m \) is an error signal in the state vector space, \( P \in \mathbb{R}^{m \times m} \) is a square matrix derived from a Lyapunov relationship and \( B \in \mathbb{R}^{m \times n} \) is the input Jacobian for the LTI system to be controlled and \( F(\Theta) = [f_1(\theta_1) \ldots f_m(\theta_m)]^T. \) The projection algorithm operates with the family of convex functions

\[ f(\theta; \vartheta, \varepsilon) = \frac{||\theta||^2 - \vartheta^2}{2\varepsilon} \]

Then, the components of the convex vector function \( F \) are chosen as

\[ f_i(\theta_i) = f(\theta_i; \vartheta_i, \varepsilon_i). \quad (10) \]

Each \( i \)-th component of \( F \) is associated with two constant scalar quantities \( \vartheta_i \) and \( \varepsilon_i. \) From (10), \( f_i(\theta_i) = 0 \) when \( ||\theta_i|| = \vartheta_i, \) and \( f_i(\theta_i) = 1 \) when \( ||\theta_i|| = \vartheta_i + \varepsilon_i. \) If the initial condition for \( \Theta \) is such that \( \Theta(t = 0) = \theta_0 = [\theta_{0,1} \ldots \theta_{0,m}] \) where \( \{\theta_{0,i}; f_i(\theta_i) \leq 0 \text{ for } i = 1 \text{ to } m\}, \) then each \( \theta_i \) satisfies all three conditions for Lemma 9. Thus \( ||\theta_i(t)|| \leq \vartheta_i + \varepsilon_i \forall t \geq 0. \)
4 $\Gamma$–Projection

**Definition 11.** A variant of the projection algorithm, $\Gamma$–projection, updates the parameter along a symmetric positive definite gain $\Gamma$ as defined below

$$
\text{Proj}_\Gamma(\theta, y, f) = \begin{cases} 
\Gamma y - \Gamma \frac{\nabla f(\theta)(\nabla f(\theta))^T}{(\nabla f(\theta))^T \Gamma \nabla f(\theta)} \Gamma y f(\theta) & \text{if } f(\theta) > 0 \land y^T \Gamma \nabla f(\theta) > 0 \\
\Gamma y & \text{otherwise}
\end{cases}
$$

This method was first introduced in [1].

**Lemma 12.** Given $\theta^* \in \Omega_0$,

$$
(\theta - \theta^*)^T (\Gamma^{-1} \text{Proj}_\Gamma(\theta, y, f) - y) \leq 0.
$$

**Proof.** If $f(\theta) > 0 \land y^T \Gamma \nabla f(\theta) > 0$, then

$$
(\theta^* - \theta)^T \left( y - \Gamma^{-1} \left( \Gamma y - \Gamma \frac{\nabla f(\theta)(\nabla f(\theta))^T}{(\nabla f(\theta))^T \Gamma \nabla f(\theta)} \Gamma y f(\theta) \right) \right)
$$

and using Lemma 4

$$
(\theta^* - \theta)^T \nabla f(\theta) (\nabla f(\theta))^T \Gamma y
$$

\[\leq 0\]

otherwise $\text{Proj}_\Gamma(\theta, y, f) = \Gamma y$. \hfill \square

**References**


