positive-definite symmetric can be made strictly positive real via constant output feedback.

Theorem 2 of this note thus completes and clarifies the implications of the formulations of Theorem 1, yet implies no argument whatsoever with [1]. Besides, [1] goes beyond previous publications and contains another important result, namely, the proof that if a system cannot be made SPR via constant feedback, no dynamic feedback can render it SPR. This note, therefore, only intends to present a brief history of early works and main developments that led to the complete proofs of this important result and do not seem to be too well known in the control community. Reference [1] served as a trigger for our note only because it was the last in a series of publications dealing with this issue.

II. BRIEF HISTORY

First, recall that a linear time-invariant system with a state-space realization \{A, B, C\} is called SPR if there exist two positive definite matrices P and Q, such that the following two relations are simultaneously satisfied:

\[ PA_K + A_K^T P = -Q \]
\[ PB = C^T. \]

The first relation shows that a SPR system is asymptotically stable. The second relation yields \( CB = B^T C^T \) (i.e., positive-definite symmetric) and implies that the transfer function \( T(s) \) has relative degree \( m \) or, in other words, \( T(s) \) has \( n \) poles and \( n - m \) zeros.

The SPR property plays a crucial role in guaranteeing stability in systems with uncertainty [2], and in adaptive control [3]. However, as most real-world systems are not inherently SPR, the result presented as theorems above has proved very useful.

Based on the SPR representation (1)–(2) one can formally define the matrix \( A = A_K + B K C \) with \( K = 0 \). It is known that the system \{A, B, C\} remains SPR if one replaces \( K = 0 \) with any positive-definite and arbitrarily large gain. If one uses a negative-definite gain instead, one gets a system that is not necessarily SPR. However, such a system only needs a constant output feedback gain to become SPR. As output feedback affects the poles yet has no effect on the zeros of the system or on its input and output matrices, one is left with a new system \{A, B, C\} that could be unstable, yet is minimum-phase and with \( C B > 0 \), like the SPR system [4]. Such systems that are separated only by constant output feedback from strict positive realness have been called “almost strictly positive real (ASPR)” [4] and many examples that satisfied the assumptions of the theorem above were shown to be ASPR and were successfully tested using adaptive control [4], [7]. Although some proofs of the ASPR property of Theorem 2 had been available in the Russian literature both for single-input–single-output (SISO) [5] and multivariable systems [6], they were not known and probably their importance not very well appreciated in the Western literature at the time. Here, some partial proofs based on simple root-locus and angle arguments have appeared in the context of SISO systems [7], [8] and the first proofs of the theorem for the general multivariable case were developed by Owens et al. [9] using multivariable root-loci and by Teixeira [10] using the state-space representation. (Teixeira also presented his proof in [11], which was submitted for publication in 1988. Unfortunately, although based on the available knowledge at the time should have been the first known state-space proof of the theorem, this paper was not published due to an oversight in the review process. A 1999 attempt by Fradkov to present a review on the topic ended with a similar result.) Various proofs of the theorem based on state-space representation have been subsequently published by Gu [12] and, as mentioned, by Huang et al. [1] as already mentioned, Huang et al. [1]
went even further in emphasizing the importance of the ASPR property, as Theorem 2 of their paper shows that if constant feedback can not make a system SPR, then there exists no proper dynamic feedback to render it SPR.

The reader may have a hard time following the proofs of the theorem, and even its general formulation and implications, as they were imbedded among many other results in the papers above, each one within the scope of its particular interests. Therefore, for the ease of reference, this note ends with a short and direct proof of Theorem 2 (which does make use of ideas used by others, yet ends with a direct and very streamlined version of the proof).

III. PROOF OF THEOREM 2

The basic ideas of the following proof actually make use of such early works as [13] and [14] that had clarified the previously obscure relationship between the state-space representation of a system and the poles and zeros of its corresponding transfer function. First recall [13] that the zeros of the closed-loop system \( \{A_K = A - BK, B, C\} \) are identical with the zeros of the open-loop system \( \{A, B, C\} \). Following [13], select the matrices \( M_{m,n}, N_{n,m}, N_{n,m,n} \) such that \( CM = 0 \), \( NB = 0 \), and \( NM = I \). Such matrices always exist [13] and the zeros of the system are the eigenvalues of the matrix \( NAM \). Because the system is strictly minimum phase, \( NAM \) is Hurwitz and there exists some positive–definite matrix \( P_0 \) such that

\[
P_0(NAM) + (NAM)^T P_0 = -Q_0 < 0.
\]

Before continuing, note that by using the relation \( NM = I \) (not used in [1]), it is easy to see that the condition formulated in Theorem 1 is equivalent with (3) and makes Theorem 1 equivalent with Theorem 2. Now, it must be shown that there exists a matrix \( K \) that fulfills the conditions of Theorem 2. To this end, consider the matrix

\[
P = N^T P_0 N + C^T (CB)^{-T} C.
\]

Although \( P \) in (4) seems to be only positive–semidefinite symmetric, it is easy to show that \( K \) is actually positive–definite symmetric. Because \( NB = 0 \) implies that the rows of \( N \) and the columns of \( B \) are linearly independent, one can select the nonsingular matrix

\[
N_H = \begin{bmatrix} N \\ B^T \end{bmatrix}
\]

so that \( P > 0 \) if and only if \( \tilde{P} = N_H P N_H^T > 0 \). Here

\[
\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix} = \begin{bmatrix} N \\ B^T \end{bmatrix} P N^T B
\]

\[
\tilde{P}_{11} = NPR_0 N^T + N C^T (CB)^{-T} C N^T
\]

\[
\tilde{P}_{12} = NPB = NC^T
\]

\[
\tilde{P}_{21} = B^T P N^T = C N^T
\]

\[
\tilde{P}_{22} = B^T N P_0 N B + B^T C^T (CB)^{-T} C B = B^T C^T.
\]

We also compute

\[
\tilde{P}_{12} \tilde{P}_{22}^{-1} \tilde{P}_{21} = N C^T (B^T C^T)^{-T} C N^T.
\]

Thus, it is clear that \( \tilde{P}_{11} > 0, \tilde{P}_{22} > 0, \) and \( \tilde{P}_{11} - \tilde{P}_{21} \tilde{P}_{22}^{-1} \tilde{P}_{21} > 0 \), which imply that \( \tilde{P} = N_H \tilde{P} N_H^T > 0 \) and therefore \( P > 0 \).

It is easy to see now that the SPR relation (2) is satisfied. However, one must also show that (2) is satisfied for some positive definite gains \( K \) “sufficiently large.” To clarify the meaning of “large” in this context, it is worth mentioning that in matrical case if constant feedback can render it SPR.

Because \( NB = 0 \) one gets

\[
P_{AK} + A_K^T P = N^T P_0 N A + A_N^T P_0 N + C^T (C B)^{-T} C A + A^T C^T (C B)^{-T} C
\]

\[
- C^T K C - C^T K^T C.
\]

Again, it is not directly seen that (13) is negative definite. Therefore, consider the matrix

\[
T = \begin{bmatrix} M & B (CB)^{-1} \end{bmatrix}.
\]

It is easy to see that \( T \) is square and invertible and its inverse is

\[
T^{-1} = \begin{bmatrix} N \\ C \end{bmatrix}.
\]

To show that (13) is negative definite, compute the expression

\[
T^T \left( PA_K + A_K^T P \right) T = -Q.
\]

Here

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}.
\]

\[
Q_{11} = -[P_0(NAM) + (NAM)^T P_0] > 0
\]

\[
Q_{12} = -P_0 N A B (CB)^{-1} - (C A N)^T (CB)^{-1}
\]

\[
Q_{21} = -(C B)^{-1} C A M - (N A B (CB)^{-1}) P_0 = Q_{12}^T
\]

\[
Q_{22} = K + K^T - (C B)^{-1} C A B (C B)^{-1}
\]

\[
- (C A B (C B)^{-1})^T (C B)^{-1}.
\]

For \( K \) sufficiently large (positive definite), one gets

\[
Q_{22} > 0
\]

and

\[
Q_{22} - Q_{21} Q_{11}^{-1} Q_{12} > 0.
\]

Relations (18), (22), and (23) show that \( Q > 0 \) and, therefore, the closed-loop system is SPR. Note that if \( K = k K_0 \), where \( k \) is scalar, there exists a limiting value \( k_0 \) for the gain \( k \) that would make \( Q \) positive semidefinite, thus defining a so-called “weakly” SPR system. This value can be considered a lower bound for the sought after gains, as any value \( k > k_0 \) would make the system (strongly) SPR.

IV. CONCLUSION

This paper presented a brief history of the main contributions to an important SPR properties that led to the simple formulation of Theorem 2. A streamlined and direct proof of the theorem has also been presented. Present research seems to show that the procedure used in
this note could eventually be extended to eliminate the symmetry condition from the SPR relations and also to extend the relations to nonstationary and nonlinear systems.

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