Adaptive Control and the Definition of Exponential Stability

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Objectives

Prove that the following statement is **incorrect**

- “If the reference model is persistently exciting then the adaptive system is globally exponentially stable”

Prove the following

- Adaptive systems can at best be uniformly asymptotically stable in the large

Main insights

- Indeed if the reference model is PE then after some time the plant will be PE, **but after exactly how much time?**
- We will show how a PE condition on the reference model implies a **weak** PE condition on the plant state.
Outline

- Definitions
  - Stability
  - Exponential Stability
  - Persistent Excitation (PE)
  - weak Persistent Excitation (PE*)

- Link between PE and Exponential Stability
- Link between PE* and Uniform Asymptotic Stability
- Simulation Studies
Uniform Stability in the Large (Global)

\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

**Solution** \( s(t; x_0, t_0) \)

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**Definition: Uniform Stability in the Large (Massera, 1956)**

(i) **Uniformly Stable:** \( \forall \epsilon > 0 \exists \delta(\epsilon) > 0 \) s.t.
\[ \|x_0\| \leq \delta \implies \|s(t; x_0, t_0)\| \leq \epsilon. \]

(ii) **Uniformly Attracting in the Large:** For all \( \rho, \eta \) \( \exists T(\eta, \rho) \)
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \eta \quad \forall t \geq t_0 + T. \]

(iii) **Uniformly Asymptotically Stable in the Large (UASL)**
\[ = \text{uniformly stable} + \text{uniformly bounded} + \text{uniformly attracting in the large}. \]
Exponential Stability

\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

**Solution** \( s(t; x_0, t_0) \)

**Definition:** (Malkin, 1935; Kalman and Bertram, 1960)

(i) **Exponentially Stable (ES):** \( \forall \rho > 0 \ \exists \ \nu(\rho), \kappa(\rho) \) s.t.

\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \kappa\|x_0\|e^{-\nu(t-t_0)} \]

(ii) **Exponentially Stable in the Large (ESL):** \( \exists \ \nu, \kappa \) s.t.

\[ \|s(t; x_0, t_0)\| \leq \kappa\|x_0\|e^{-\nu(t-t_0)} \]
Persistent Excitation

“Exogenous Signal” : \( \omega : [t_0, \infty) \rightarrow \mathbb{R}^p \)

Initial Condition : \( \omega_0 = \omega(t_0) \)

Parameterized Function : \( y(t, \omega) : [t_0, \infty) \times \mathbb{R}^p \rightarrow \mathbb{R}^m \)

**Definition**

(i) **Persistently Exciting (PE):**

\[ \exists T, \alpha \text{ s.t.} \]
\[ \int_0^{t+T} y(\tau, \omega)y^T(\tau, \omega) d\tau \geq \alpha I \]

for all \( t \geq t_0 \) and \( \omega_0 \in \mathbb{R}^p \).

(ii) **weakly Persistently Exciting (PE*) (\( \omega, \Omega \)):**

\[ \exists \text{ a compact set } \Omega \subset \mathbb{R}^p, \quad T(\Omega) > 0, \alpha(\Omega) \text{ s.t.} \]
\[ \int_0^{t+T} y(\tau, \omega)y^T(\tau, \omega) d\tau \geq \alpha I \]

for all \( \omega_0 \in \Omega \) and \( t \geq t_0 \).
properties of adaptive control
Adaptive Control

Plant: \[ \dot{x} = Ax - B\theta^T x + Bu \]

Reference Model: \[ \dot{x}_m = Ax_m + Br \]

Control Input: \[ u = \hat{\theta}^T(t)x + r \]

Error: \[ e = x - x_m \]

Parameter Error: \[ \tilde{\theta}(t) = \hat{\theta}(t) - \theta \]

Update Law: \[ \dot{\hat{\theta}}(t) = -xe^TPB \]

Stability: \[ V(e(t), \tilde{\theta}(t)) = e^T(t)Pe(t) + \text{Trace}\left(\tilde{\theta}^T(t)\tilde{\theta}(t)\right) \]

\[ \dot{V} \leq e^TQe \]

\[ \|e\|_{L_{\infty}} \leq \sqrt{V(e(t_0), \tilde{\theta}(t_0))/P_{\min}} \]

\[ \|e\|_{L_2} \leq \sqrt{V(e(t_0), \tilde{\theta}(t_0))/Q_{\min}} \]

The L-norms of \(e\) are initial condition dependent!!
Exponential Stability and Adaptive Control

\[ \dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^TP & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \hat{\theta}(t) \end{bmatrix} \]

**Theorem:** (Morgan and Narendra, 1977)

If \( x(t) \in \text{PE} \) then \( z(t) = 0 \) is UASL.

Therefore, when \( x \in \text{PE} \) the dynamics \( z(t) \) are globally exponentially stable (Anderson, 1977).

The condition of PE for \( x(t) \) however does not follow from \( x_m(t) \in \text{PE} \).
If $x_m \in \text{PE}$ then $x \in \?$

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

$$
(x_m^T h)^2 - (x^T h)^2 = -(x^T h - x_m^T h) (x^T h + x_m^T h) \\
\leq \|e\| \quad = e^T h + 2x_m^T h
$$

$$(x_m^T h)^2 - (x^T h)^2 \leq \|e\| \left( \sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max} \right)
$$

Move $x_m$ to the RHS, multiply by $-1$, and integrate to $pT$

$$
\int_t^{t+pT} (x^T(\tau) h)^2 d\tau \geq \\
p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max} \right) \sqrt{pT \int_t^{t+pT} \|e(\tau)\|^2 d\tau}.
$$

Clean the notation

$$
\int_t^{t+pT} \|x\|^2 d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max} \right) \sqrt{pT \frac{V(z_0)}{Q_{\min}}}.
$$
\[ x \in \text{PE}^* \quad x \notin \text{PE} \]

\[
\int_{t}^{t+T} x_m(\tau)x_m^T(\tau)d\tau \geq \alpha I
\]

\[
\int_{t}^{t+pT} x^T(\tau, z)x(\tau, z)d\tau \geq p\alpha - \left(\sqrt{\frac{V(z_0)}{P_{min}}} + 2x_{m}^{\text{max}}\right) \sqrt{pT\frac{V(z_0)}{Q_{min}}}.
\]

**Fixed** \( T, \alpha \) \quad **Free** \( p \) \quad **Initial Condition** \( z_0 \)

If the initial condition \( \|z(t_0)\| \) increases (\( V(z_0) \) increases), then \( p \) must increase, and thus the time (\( pT \)) must increase to keep \( \alpha' \) constant.

Revisit the definitions for PE

(i) **Persistently Exciting** (PE): \( \exists T, \alpha \) s.t.

\[
\int_{t}^{t+T} x(\tau, \omega)x^T(\tau, \omega)d\tau \geq \alpha I
\]

for all \( t \geq t_0 \) and \( \omega_0 \in \mathbb{R}^p \).

(ii) **weakly Persistently Exciting** (PE\(^*\)\((\omega, \Omega)\)): \( \exists \) a compact set \( \Omega \subset \mathbb{R}^p, \ T(\Omega) > 0, \ \alpha(\Omega) \) s.t.

\[
\int_{t}^{t+T} x(\tau, \omega)x^T(\tau, \omega)d\tau \geq \alpha I
\]

for all \( \omega_0 \in \Omega \) and \( t \geq t_0 \).
Adaptive Control and UASL

Revisit the adaptive control problem

\[
\dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^T & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix}
\]

Define the following compact set

\[
\Omega(\zeta) \triangleq \{ z : V(z) \leq \zeta \}
\]

**Theorem**

If \( x_m \in \text{PE} \) then \( x \in \text{PE}^*(z, \Omega(\zeta)) \), for any \( \zeta > 0 \), and it follows that the dynamics above are UASL.

**Proof.**

- \( x_m \in \text{PE} \implies x \in \text{PE}^*(z, \Omega(\zeta)) \) from previous slide.
- \( \text{PE}^* \) by definition is a local uniform property
- The “Large” part of UASL holds because we can take arbitrarily large \( \Omega \)
- Next we prove (by counter example) \( x_m \in \text{PE} \) does not imply ESL.
Simulation Example

Plant: \[ \dot{x} = Ax - B\theta^T x + Bu \]

Reference: \[ \dot{x}_m = Ax_m + Br \]

Control: \[ u = \hat{\theta}^T(t)x + r \]

- \[ \hat{\theta} = x_m(t_0) = 3 \]
- \[ A = -1 \]
- \[ B = 1 \]
- \[ r = 3 \]

\[ M_1 \cup M_2 \cup M_3 \text{ is invariant} \]

\[ M_3 \text{ extends down in an unbounded fashion} \]

\[ \text{maximum rate of change in } M_3 \text{ is bounded} \]

\[ \text{The fixed rate regardless of initial condition implies that ESL is impossible} \]

\( (\text{Jenkins et al., 2013a; 2013b}) \)

Summary

- PE of the reference model does not imply PE for the state vector
- Adaptive control in general can not be guaranteed to be ESL

Bibliography


backup slides
Stability

\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

**Solution** \( s(t; x_0, t_0) \)

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**Definition: Stability** (Massera, 1956)

(i) **Stable:** \( \forall \varepsilon > 0 \exists \delta(\varepsilon, x_0, t_0) > 0 \) s.t.
\[ \|x_0\| \leq \delta \implies \|s(t; x_0, t_0)\| \leq \varepsilon. \]

(ii) **Attracting:** \( \exists \rho(t_0) > 0 \) s.t. \( \forall \eta > 0 \exists \) an attraction time \( T(\eta, x_0, t_0) \) s.t.
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \eta \quad \forall t \geq t_0 + T. \]

(iii) **Asymptotically Stable** = stable + attracting.
\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

**Solution** \( s(t; x_0, t_0) \)

**Definition: **Uniform Stability (Massera, 1956)

**(iv) Uniformly Stable:** \( \delta(\epsilon) \) in (i) is uniform in \( t_0 \) and \( x_0 \).

**(v) Uniformly Attracting:** \( \rho \) and \( T \) do not depend on \( t_0 \) or \( x_0 \) and thus the attracting times take the form \( T(\eta, \rho) \).

**(vi) Uniformly Asymptotically Stable, (UAS) \( = \) uniformly stable + uniformly attracting.**
Uniform Stability in the Large (Global)

\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

**Solution** \( s(t; x_0, t_0) \)

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**Definition: Uniform Stability in the Large** (Massera, 1956)

(vii) **Uniformly Attracting in the Large**: For all \( \rho, \eta \) \( \exists T(\eta, \rho) \)
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \eta \quad \forall t \geq t_0 + T. \]

(viii) **Uniformly Asymptotically Stable in the Large (UASL)**
= uniformly stable +
uniformly bounded +
uniformly attracting in the large.
Exponential Asymptotic Stability

\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

**Solution** \( s(t; x_0, t_0) \)

**Definition:** (Malkin, 1935; Kalman and Bertram, 1960)

(i) **Exponentially Asymptotically Stable (EAS):**
\[ \forall \, \epsilon > 0 \, \exists \, \delta(\epsilon), \nu(\epsilon) \text{ s.t.} \]
\[ \|x_0\| \leq \delta \implies \|s(t; x_0, t_0)\| \leq \epsilon e^{-\nu(t-t_0)} \]

(ii) **Exponentially Asymptotically Stable in the Large (EASL):** \( \forall \, \rho > 0 \, \exists \, \epsilon(\rho), \nu(\rho) \) s.t.
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \epsilon e^{-\nu(t-t_0)} \]

(iii) **Exponentially Stable (ES):** \( \forall \, \rho > 0 \, \exists \, \nu(\rho), \kappa(\rho) \) s.t.
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \kappa \|x_0\| e^{-\nu(t-t_0)} \]

(iv) **Exponentially Stable in the Large (ESL):** \( \exists \nu, \kappa \) s.t.
\[ \|s(t; x_0, t_0)\| \leq \kappa \|x_0\| e^{-\nu(t-t_0)} \]