Adaptive Control and the Definition of Exponential Stability

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Objectives

Prove that the following statement is incorrect:

- “If the reference model is persistently exciting then the adaptive system is globally exponentially stable”
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Prove that the following statement is **incorrect**

- “If the reference model is persistently exciting then the adaptive system is globally exponentially stable”

Prove the following

- Adaptive systems can at best be uniformly asymptotically stable in the large
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Prove that the following statement is incorrect
- “If the reference model is persistently exciting then the adaptive system is globally exponentially stable”

Prove the following
- Adaptive systems can at best be uniformly asymptotically stable in the large

Main insights
- Indeed if the reference model is PE then after some time the plant will be PE, but after exactly how much time?
- We will show how a PE condition on the reference model implies a weak PE condition on the plant state.
Outline

- Definitions
  - Stability
  - Exponential Stability
  - Persistent Excitation (PE)
  - weak Persistent Excitation (PE*)

- Link between PE and Exponential Stability
- Link between PE* and Uniform Asymptotic Stability
- Simulation Studies
Uniform Stability in the Large (Global)

\[
\dot{x}(t) = f(x(t), t)
\]

\[x_0 \triangleq x(t_0)\]

**Solution** \(s(t; x_0, t_0)\)
Uniform Stability in the Large (Global)

\[ \dot{x}(t) = f(x(t), t) \]

\( x_0 \triangleq x(t_0) \)

Solution \( s(t; x_0, t_0) \)

**Definition: Uniform Stability in the Large** (Massera, 1956)

(i) Uniformly Stable: \( \forall \epsilon > 0 \ \exists \ \delta(\epsilon) > 0 \ \text{s.t.} \)

\[ \|x_0\| \leq \delta \implies \|s(t; x_0, t_0)\| \leq \epsilon. \]
Uniform Stability in the Large (Global)

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(ii) **Uniformly Attracting in the Large:** For all \( \rho, \eta \exists T(\eta, \rho) \)

\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \eta \quad \forall t \geq t_0 + T. \]
Uniform Stability in the Large (Global)

\[ \dot{x}(t) = f(x(t), t) \]
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(iii) **Uniformly Asymptotically Stable in the Large (UASL)**
\[ = \text{uniformly stable} + \text{uniformly bounded} + \text{uniformly attracting in the large.} \]
Exponential Stability

\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

Solution \( s(t; x_0, t_0) \)

Definition: (Malkin, 1935; Kalman and Bertram, 1960)

(i) **Exponentially Stable (ES):** \( \forall \rho > 0 \exists \nu(\rho), \kappa(\rho) \) s.t.

\[
\|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \kappa \|x_0\| e^{-\nu(t-t_0)}
\]
Exponential Stability

\[ \dot{x}(t) = f(x(t), t) \]
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**Solution** \( s(t; x_0, t_0) \)

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(ii) **Exponentially Stable in the Large (ESL):** \( \exists \ \nu, \kappa \) s.t.

\[ \|s(t; x_0, t_0)\| \leq \kappa \|x_0\| e^{-\nu(t-t_0)} \]
Persistent Excitation

“Exogenous Signal”: \( \omega : [t_0, \infty) \to \mathbb{R}^p \)

Initial Condition: \( \omega_0 = \omega(t_0) \)

Parameterized Function: \( y(t, \omega) : [t_0, \infty) \times \mathbb{R}^p \to \mathbb{R}^m \)
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**Definition**

(i) **Persistently Exciting (PE):**

\[ \exists \ T, \alpha \text{ s.t.} \]

\[ \int_t^{t+T} y(\tau, \omega)y^\top(\tau, \omega) d\tau \geq \alpha I \]

for all \( t \geq t_0 \) and \( \omega_0 \in \mathbb{R}^p \).
Persistent Excitation

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**Definition**

(i) **Persistently Exciting** (PE):
\[
\exists \ T, \alpha \text{ s.t. } \int_{t}^{t+T} y(\tau, \omega)y^T(\tau, \omega)d\tau \geq \alpha I
\]

for all \( t \geq t_0 \) and \( \omega_0 \in \mathbb{R}^p \).

(ii) **weakly Persistently Exciting** (PE\(^*\)(\(\omega, \Omega\))):
\[
\exists \ a \text{ compact set } \Omega \subset \mathbb{R}^p, \ T(\Omega) > 0, \alpha(\Omega) \text{ s.t. } \int_{t}^{t+T} y(\tau, \omega)y^T(\tau, \omega)d\tau \geq \alpha I
\]

for all \( \omega_0 \in \Omega \) and \( t \geq t_0 \).
properties of adaptive control
Adaptive Control

Plant
\[ \dot{x} = A x - B \theta^T x + B u \]

Reference Model
\[ \dot{x}_m = A x_m + B r \]
Adaptive Control

Plant
\[ \dot{x} = Ax - B \theta^T x + Bu \]

Reference Model
\[ \dot{x}_m = Ax_m + Br \]

Unknown Parameter \( \theta \)
Adaptive Control

Plant: \[
\dot{x} = Ax - B\theta^T x + Bu
\]

Reference Model: \[
\dot{x}_m = Ax_m + Br
\]

Control Input: \[
u = \hat{\theta}^T(t)x + r
\]

Unknown Parameter \(\theta\)

Adaptive Parameter \(\hat{\theta}(t)\)
Adaptive Control

Plant: \[ \dot{x} = Ax - B\theta^T x + Bu \]

Reference Model: \[ \dot{x}_m = Ax_m + Br \]

Control Input: \[ u = \hat{\theta}^T(t)x + r \]

Error: \[ e = x - x_m \]

Parameter Error: \[ \tilde{\theta}(t) = \hat{\theta}(t) - \theta \]

Diagram:
- Reference Model: \[ x_m \]
- Plant: \[ x \]
- Control Input: \[ u \]
- Error: \[ e \]
- Unknown Parameter: \[ \theta \]
- Adaptive Parameter: \[ \hat{\theta}(t) \]
Adaptive Control

Plant \[ \dot{x} = Ax - B\theta^T x + Bu \]
Reference Model \[ \dot{x}_m = Ax_m + Br \]
Control Input \[ u = \hat{\theta}^T(t)x + r \]
Error \[ e = x - x_m \]
Parameter Error \[ \tilde{\theta}(t) = \hat{\theta}(t) - \theta \]
Update Law \[ \hat{\theta}(t) = -xe^TPB \]
Adaptive Control

**Plant**
\[ \dot{x} = Ax - B\theta^T x + Bu \]

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\[ \dot{x}_m = Ax_m + Br \]

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\[ u = \hat{\theta}^T(t)x + r \]

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\[ e = x - x_m \]

**Parameter Error**
\[ \tilde{\theta}(t) = \hat{\theta}(t) - \theta \]

**Update Law**
\[ \dot{\hat{\theta}}(t) = -xe^TPB \]

**Stability**
\[ V(e(t), \tilde{\theta}(t)) = e^T(t)Pe(t) + \text{Trace} \left( \tilde{\theta}^T(t)\tilde{\theta}(t) \right) \]
Adaptive Control

Plant: \[ \dot{x} = Ax - B\theta^T x + Bu \]
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\[ \dot{V} \leq e^TQe \]
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\[ V(e(t), \tilde{\theta}(t)) = e^T(t)Pe(t) + \text{Trace} \left( \tilde{\theta}^T(t)\tilde{\theta}(t) \right) \]
\[ \dot{V} \leq e^TQe \]
\[ \|e\|_{L_\infty} \leq \sqrt{V(e(t_0), \tilde{\theta}(t_0))/P_{\text{min}}} \]
\[ \|e\|_{L_2} \leq \sqrt{V(e(t_0), \tilde{\theta}(t_0))/Q_{\text{min}}} \]
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Plant \[ \dot{x} = Ax - B\theta^T x + Bu \]

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The L-norms of $e$ are initial condition dependent!!
Exponential Stability and Adaptive Control

\[ \dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^T & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix} \]
Exponential Stability and Adaptive Control

\[ \dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^TP & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix} \]

**Theorem:** (Morgan and Narendra, 1977)

If \( x(t) \in \text{PE} \) then \( z(t) = 0 \) is UASL.
Exponential Stability and Adaptive Control

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\dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^TP & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix}
\]

**Theorem:** (Morgan and Narendra, 1977)

If \( x(t) \in PE \) then \( z(t) = 0 \) is UASL.

![Diagram showing relationships between ESL, ES, UASL, and UAS with a linear component.](image)
Exponential Stability and Adaptive Control

\[ \dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^TP & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix} \]

**Theorem:** (Morgan and Narendra, 1977)

If \( x(t) \in \text{PE} \) then \( z(t) = 0 \) is UASL.

Therefore, when \( x \in \text{PE} \) the dynamics \( z(t) \) are globally exponentially stable (Anderson, 1977).
Exponential Stability and Adaptive Control

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\dot{z}(t) = \begin{bmatrix}
A & Bx^T(t) \\
-x(t)B^TP & 0
\end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix}
e(t) \\ \tilde{\theta}(t)
\end{bmatrix}
\]

**Theorem:** (Morgan and Narendra, 1977)

If \( x(t) \in \text{PE} \) then \( z(t) = 0 \) is UASL.

Therefore, when \( x \in \text{PE} \) the dynamics \( z(t) \) are globally exponentially stable (Anderson, 1977).

The condition of PE for \( x(t) \) however does not follow from \( x_m(t) \in \text{PE} \).
If $x_m \in \text{PE}$ then $x \in \mathbb{R}$?

Recall that $e = x - x_m$, then for any fixed unitary vector $h$
If $x_m \in \text{PE}$ then $x \in \text{?}$

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

$$(x_m^T h)^2 - (x^T h)^2 = - (x^T h - x_m^T h)(x^T h + x_m^T h)$$
If $x_m \in \text{PE}$ then $x \in \ ?$

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

\[
(x_m^T h)^2 - (x^T h)^2 = -(x^T h - x_m^T h) (x^T h + x_m^T h) \leq \|e\| = e^T h + 2 x_m^T h
\]
If $x_m \in \text{PE}$ then $x \in \mathbb{R}$?

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

\[
(x_m h)^2 - (x h)^2 = -\left( (x^T h - x_m^T h) (x^T h + x_m^T h) \right) 
\leq \|e\| \quad (= e^T h + 2x_m^T h)
\]

\[
(x_m h)^2 - (x h)^2 \leq \|e\| \left( \sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max} \right)
\]
If $x_m \in \text{PE}$ then $x \in ?$

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

$$
(x_m h) \dot{x} - (x h) \dot{x} = -(x^T h - x_m^T h)(x^T h + x_m^T h) \\
\leq \|e\| (x_m^T h + 2x_m^T h) = e^T h + 2x_m^T h
$$

$$
(x_m h)^T (x_m h) - (x h)^T (x h) \leq \|e\| \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^{\text{max}} \right)
$$
If $x_m \in \text{PE}$ then $x \in ?$

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(x_m h)^2 - (x^T h)^2 = -\left( x^T h - x_m^T h \right) \left( x^T h + x_m^T h \right) \leq \|e\| \leq \|e\| + 2x_m^T h
\]

\[
(x_m h)^2 - (x^T h)^2 \leq \|e\| \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^\text{max} \right)
\]

\[
\|e\|_{L_\infty} \leq \sqrt{\frac{V(z_0)}{P_{\text{min}}}}
\]
If $x_m \in PE$ then $x \in \Omega$?

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

$$(x_m^T h)^2 - (x^T h)^2 = -(x^T h - x_m^T h) (x^T h + x_m^T h) \leq e^T h + 2 x_m^T h$$

$$(x_m^T h)^2 - (x^T h)^2 \leq \|e\| \left( \sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max} \right)$$

Move $x_m$ to the RHS, multiply by $-1$, and integrate to $p^T$

$$\|e\|_{L_\infty} \leq \sqrt{\frac{V(z_0)}{P_{\min}}}$$
If \( x_m \in \text{PE} \) then \( x \in ? \)

Recall that \( e = x - x_m \), then for any fixed unitary vector \( h \)

\[
(x_m^T h)^2 - (x^T h)^2 = -(x^T h - x_m^T h)(x^T h + x_m^T h) \\
\leq \|e\| \leq e^T h + 2x_m^T h
\]

\[
(x_m^T h)^2 - (x^T h)^2 \leq \|e\| \left( \sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\text{max}} \right)
\]

Move \( x_m \) to the RHS, multiply by \(-1\), and integrate to \( pT \)

\[
\int_t^{t+pT} (x^T(\tau)h)^2 d\tau \geq \\
p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\text{max}} \right) \sqrt{pT \int_t^{t+pT} \|e(\tau)\|^2 d\tau}.
\]
If $x_m \in \text{PE}$ then $x \in ?$

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Move $x_m$ to the RHS, multiply by $-1$, and integrate to $pT$

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p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^{\text{max}} \right) \sqrt{pT \int_t^{t+pT} \|e(\tau)\|^2 d\tau}.
\]
If $x_m \in PE$ then $x \in \bigcirc$?

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

$$
(x_m^T h)^2 - (x^T h)^2 = \left(\begin{array}{c}
\leq \|e\| \\
e^{T} h + 2x_m^T h
\end{array}\right)
$$

$$
(x_m^T h)^2 - (x^T h)^2 \leq \|e\| \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max}\right)
$$

Move $x_m$ to the RHS, multiply by $-1$, and integrate to $pT$

$$
\int_t^{t+pT} (x^T (\tau) h)^2 d\tau \geq \int_{t_0}^{t_0+T} x_m x_m^T \geq \alpha I
$$

$$
p\alpha - \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max}\right) \sqrt{pT \int_t^{t+pT} \|e(\tau)\|^2 d\tau}.
$$
If $x_m \in PE$ then $x \in \mathbb{R}$?

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

$$(x_m^T h)^2 - (x^T h)^2 = -(x^T h - x_m^T h)(x^T h + x_m^T h) \leq \|e\| \underbrace{x^T h + 2 x_m^T h}_\text{=}$$

$$(x_m^T h)^2 - (x^T h)^2 \leq \|e\| \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^\text{max} \right)$$

Move $x_m$ to the RHS, multiply by $-1$, and integrate to $pT$

$$\int_t^{t+pT} (x^T(\tau) h)^2 d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^\text{max} \right) \sqrt{pT \int_t^{t+pT} \|e(\tau)\|^2 d\tau}.$$
If $x_m \in \text{PE}$ then $x \in \text{?}$

Recall that $e = x - x_m$, then for any fixed unitary vector $h$

\[
(x_m^T h)^2 - (x^T h)^2 = -(x^T h - x_m^T h) (x^T h + x_m^T h) \leq \|e\| = e^T h + 2x_m^T h
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(x_m^T h)^2 - (x^T h)^2 \leq \|e\| \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^{\max} \right)
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Move $x_m$ to the RHS, multiply by $-1$, and integrate to $pT$

\[
\int_t^{t+pT} (x^T(\tau)h)^2 d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^{\max} \right) \sqrt{pT \int_t^{t+pT} \|e(\tau)\|^2 d\tau}.
\]

Clean the notation

\[
\int_t^{t+pT} \|x\|^2 d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^{\max} \right) \sqrt{pT \frac{V(z_0)}{Q_{\text{min}}}}.
\]
\( x \in \text{PE}^* \quad x \notin \text{PE} \)

\[
\int_t^{t+T} x_m(\tau)x_m(\tau)d\tau \geq \alpha I
\]

\[
\int_t^{t+pT} x^T(\tau, z)x(\tau, z)d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_{\text{max}} \right) \sqrt{pT \frac{V(z_0)}{Q_{\text{min}}}}.
\]
\[ x \in \text{PE}^* \quad \overline{x} \notin \text{PE} \]

\[
\int_t^{t+T} x_m(\tau)x_m^T(\tau)d\tau \geq \alpha I
\]

\[
\int_t^{t+p^T} x^T(\tau, z)x(\tau, z)d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_{\text{max}}^{\text{max}} \right) \sqrt{pT \frac{V(z_0)}{Q_{\text{min}}}}.
\]

Fixed \( T, \alpha \)
\[ x \in \text{PE}^* \quad x \not\in \text{PE} \]

\[
\int_t^{t+T} x_m(\tau)x_m^\top(\tau)d\tau \geq \alpha I
\]

\[
\int_t^{t+pT} x^\top(\tau, z)x(\tau, z)d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_m^{\text{max}} \right) \sqrt{pT \frac{V(z_0)}{Q_{\text{min}}}}.
\]

**Fixed** T, α  **Free** p
\[ x \in \text{PE}^* \quad x \notin \text{PE} \]

\[
\int_{t}^{t+T} x_m(\tau)x_m(\tau) d\tau \geq \alpha I
\]

\[
\int_{t}^{t+pT} x^T(\tau, z)x(\tau, z) d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_{m}^{\text{max}} \right) \sqrt{pT \frac{V(z_0)}{Q_{\text{min}}}}.
\]

Fixed \( T, \alpha \) \quad Free \ p \quad Initial Condition \( z_0 \)
\[ x \in \text{PE}^* \quad x \notin \text{PE} \]

\[
\int_{t}^{t+T} x_m(\tau)x_m^T(\tau)d\tau \geq \alpha I
\]

\[
\int_{t}^{t+pT} x^T(\tau, z)x(\tau, z)d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\text{min}}}} + 2x_{m}^{\text{max}} \right) \sqrt{pT\frac{V(z_0)}{Q_{\text{min}}}}.
\]

**Fixed** \( T, \alpha \quad \text{Free} \ p \quad \text{Initial Condition} \ z_0

If the initial condition \( \|z(t_0)\| \) increases (\( V(z_0) \) increases), then \( p \) must increase, and thus the time (\( pT \)) must increase to keep \( \alpha' \) constant.
\[ x \in \text{PE}^* \quad x \notin \text{PE} \]

\[
\int_{t}^{t+T} x_m(\tau)x_m^T(\tau)d\tau \geq \alpha I
\]

\[
\int_{t}^{t+pT} x^T(\tau, z)x(\tau, z)d\tau \geq p\alpha - \left( \sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_{m}^{\max} \right) \sqrt{pT \frac{V(z_0)}{Q_{\min}}}.
\]

**Fixed** \( T, \alpha \quad \text{Free} \ p \quad \text{Initial Condition} \ z_0 \)

If the initial condition \( \|z(t_0)\| \) increases (\( V(z_0) \) increases), then \( p \) must increase, and thus the time (\( pT \)) must increase to keep \( \alpha' \) constant.

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**Revisit the definitions for PE**

(i) **Persistently Exciting** \( (\text{PE}) \): \( \exists \ T, \alpha \) s.t.

\[
\int_{t}^{t+T} x(\tau, \omega)x^T(\tau, \omega)d\tau \geq \alpha I
\]

for all \( t \geq t_0 \) and \( \omega_0 \in \mathbb{R}^p \).

(ii) **weakly Persistently Exciting** \( (\text{PE}^*(\omega, \Omega)) \): \( \exists \) a compact set \( \Omega \subset \mathbb{R}^p \), \( T(\Omega) > 0 \), \( \alpha(\Omega) \) s.t.

\[
\int_{t}^{t+T} x(\tau, \omega)x^T(\tau, \omega)d\tau \geq \alpha I
\]

for all \( \omega_0 \in \Omega \) and \( t \geq t_0 \).
Adaptive Control and UASL

Revisit the adaptive control problem

\[ \dot{z}(t) = \begin{bmatrix} A & Bx^\top(t) \\ -x(t)B^\top P & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix} \]
Adaptive Control and UASL

Revisit the adaptive control problem

\[ \dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^TP & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix} \]

Define the following compact set

\[ \Omega(\zeta) \triangleq \{ z : V(z) \leq \zeta \} \]
Adaptive Control and UASL

Revisit the adaptive control problem

\[
\dot{z}(t) = \begin{bmatrix}
  A & Bx^T(t) \\
  -x(t)B^TP & 0
\end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix}
  e(t) \\
  \tilde{\theta}(t)
\end{bmatrix}
\]

Define the following compact set

\[\Omega(\zeta) \triangleq \{z : V(z) \leq \zeta\}\]

**Theorem**

If \(x_m \in \text{PE}\) then \(x \in \text{PE}^*(z, \Omega(\zeta))\), for any \(\zeta > 0\), and it follows that the dynamics above are UASL.
Adaptive Control and UASL

Revisit the adaptive control problem

\[ \dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^TP & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix} \]

Define the following compact set

\[ \Omega(\zeta) \triangleq \{ z : V(z) \leq \zeta \} \]

**Theorem**

If \( x_m \in \text{PE} \) then \( x \in \text{PE}^*(z, \Omega(\zeta)) \), for any \( \zeta > 0 \), and it follows that the dynamics above are UASL.

**Proof.**

\[ x_m \in \text{PE} \implies x \in \text{PE}^*(z, \Omega(\zeta)) \text{ from previous slide.} \]
Adaptive Control and UASL

Revisit the adaptive control problem

$$\dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^TP & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix}$$

Define the following compact set

$$\Omega(\zeta) \triangleq \{ z : V(z) \leq \zeta \}$$

**Theorem**

If $x_m \in \text{PE}$ then $x \in \text{PE}^*(z, \Omega(\zeta))$, for any $\zeta > 0$, and it follows that the dynamics above are UASL.

**Proof.**

- $x_m \in \text{PE} \implies x \in \text{PE}^*(z, \Omega(\zeta))$ from previous slide.
- $\text{PE}^*$ by definition is a local uniform property
Adaptive Control and UASL

Revisit the adaptive control problem

\[
\dot{z}(t) = \begin{bmatrix}
A & Bx^T(t) \\
-x(t)B^TP & 0
\end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix}
e(t) \\
\tilde{\theta}(t)
\end{bmatrix}
\]

Define the following compact set

\[
\Omega(\zeta) \triangleq \{z : V(z) \leq \zeta\}
\]

**Theorem**

If \( x_m \in PE \) then \( x \in PE^*(z, \Omega(\zeta)) \), for any \( \zeta > 0 \), and it follows that the dynamics above are UASL.

**Proof.**

- \( x_m \in PE \implies x \in PE^*(z, \Omega(\zeta)) \) from previous slide.
- \( PE^* \) by definition is a local uniform property
- The “Large” part of UASL holds because we can take arbitrarily large \( \Omega \)
Adaptive Control and UASL

Revisit the adaptive control problem

\[ \dot{z}(t) = \begin{bmatrix} A & Bx^T(t) \\ -x(t)B^T & 0 \end{bmatrix} z(t), \quad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix} \]

Define the following compact set

\[ \Omega(\zeta) \triangleq \{ z : V(z) \leq \zeta \} \]

**Theorem**

If \( x_m \in \text{PE} \) then \( x \in \text{PE}^*(z, \Omega(\zeta)) \), for any \( \zeta > 0 \), and it follows that the dynamics above are UASL.

**Proof.**

- \( x_m \in \text{PE} \implies x \in \text{PE}^*(z, \Omega(\zeta)) \) from previous slide.
- \( \text{PE}^* \) by definition is a local uniform property
- The “Large” part of UASL holds because we can take arbitrarily large \( \Omega \)
- Next we prove (by counter example) \( x_m \in \text{PE} \) does not imply ESL.
Simulation Example

Plant \[ \dot{x} = Ax - B\theta^T x + Bu \]
Reference \[ \dot{x}_m = Ax_m + Br \]
Control \[ u = \hat{\theta}^T(t)x + r \]

\[ A = -1 \]
\[ B = 1 \]
\[ r = 3 \]
\[ x_m(t_0) = 3 \]
Simulation Example

Plant $\dot{x} = Ax - B\theta^T x + Bu$

Reference $\dot{x}_m = Ax_m + Br$

Control $u = \hat{\theta}^T(t)x + r$

$A = -1$

$B = 1$

$r = 3$

$x_m(t_0) = 3$

(Jenkins et al., 2013a; 2013b)
Simulation Example

![Graph showing simulation example](image)

Plant: \( \dot{x} = Ax - B\theta^T x + Bu \)

Reference: \( \dot{x}_m = Ax_m + Br \)

Control: \( u = \hat{\theta}^T(t)x + r \)

- \( A = -1 \)
- \( B = 1 \)
- \( r = 3 \)
- \( x_m(t_0) = 3 \)

\( M_1 \cup M_2 \cup M_3 \) is invariant (Jenkins et al., 2013a; 2013b)
Simulation Example

Plant \[
\dot{x} = Ax - B\theta^T x + Bu
\]
Reference \[
\dot{x}_m = Ax_m + Br
\]
Control \[
u = \hat{\theta}^T(t)x + r
\]

- \[A = -1\]
- \[B = 1\]
- \[r = 3\]
- \[x_m(t_0) = 3\]

- \(M_1 \cup M_2 \cup M_3\) is invariant
- \(M_3\) extends down in an unbounded fashion

(Jenkins et al., 2013a; 2013b)
Simulation Example

Plant: \[ \dot{x} = Ax - B\theta^T x + Bu \]
Reference: \[ \dot{x}_m = Ax_m + Br \]
Control: \[ u = \hat{\theta}^T(t)x + r \]

\[
A = -1 \\
B = 1 \\
r = 3 \\
x_m(t_0) = 3
\]

- \( M_1 \cup M_2 \cup M_3 \) is invariant
- \( M_3 \) extends down in an unbounded fashion
- maximum rate of change in \( M_3 \) is bounded

(Jenkins et al., 2013a; 2013b)
Simulation Example

Plant  \[ \dot{x} = Ax - B\theta^T x + Bu \]
Reference  \[ \dot{x}_m = Ax_m + Br \]
Control  \[ u = \hat{\theta}^T(t)x + r \]

\[ A = -1 \]
\[ B = 1 \]
\[ r = 3 \]
\[ x_m(t_0) = 3 \]

- \( M_1 \cup M_2 \cup M_3 \) is invariant
- \( M_3 \) extends down in an unbounded fashion
- maximum rate of change in \( M_3 \) is bounded
- The fixed rate regardless of initial condition implies that ESL is impossible

(Jenkins et al., 2013a; 2013b)
Simulation Example Continued


Simulation Example Continued


Summary

- PE of the reference model does not imply PE for the state vector
- Adaptive control in general can not be guaranteed to be ESL

Bibliography


backup slides
Theorem 1: Consider the system in (1), the control law of (3), and let $p \geq n$ be the number of recorded data points. Let $X_k = [x_1, x_2, \ldots, x_p]$ be the history stack matrix containing recorded states, and $R_k = [r_1, r_2, \ldots, r_p]$ be the history stack matrix containing recorded reference signals. Assume that over a finite interval $[0, T]$ the exogenous reference input $r(t)$ is exciting, the history stack matrices are empty at $t = 0$, and are consequently updated using Algorithm 1 of [20]. Then, the concurrent learning weight update laws of (7) and (8) guarantee that the zero solution $(e(t), \tilde{K}(t), \tilde{K}_r(t)) \equiv 0$ is globally exponentially stable.
Stability

\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

**Solution** \( s(t; x_0, t_0) \)

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**Definition: Stability** (Massera, 1956)

(i) **Stable**: \( \forall \, \epsilon > 0 \, \exists \, \delta(\epsilon, x_0, t_0) > 0 \) s.t.
\[ \|x_0\| \leq \delta \implies \|s(t; x_0, t_0)\| \leq \epsilon. \]

(ii) **Attracting**: \( \exists \, \rho(t_0) > 0 \) s.t. \( \forall \, \eta > 0 \, \exists \, \text{an attraction time} \, T(\eta, x_0, t_0) \) s.t.
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \eta \, \forall \, t \geq t_0 + T. \]

(iii) **Asymptotically Stable** = stable + attracting.
$\dot{x}(t) = f(x(t), t)$

$x_0 \triangleq x(t_0)$

**Solution** $s(t; x_0, t_0)$

---

**Definition: Uniform Stability (Massera, 1956)**

1. **Uniformly Stable**: $\delta(\epsilon)$ in (i) is uniform in $t_0$ and $x_0$.
2. **Uniformly Attracting**: $\rho$ and $T$ do not depend on $t_0$ or $x_0$ and thus the attracting times take the form $T(\eta, \rho)$.
Uniform Stability in the Large (Global)

\[ \dot{x}(t) = f(x(t), t) \]
\[ x_0 \triangleq x(t_0) \]

**Solution** \( s(t; x_0, t_0) \)

**Definition: Uniform Stability in the Large** (Massera, 1956)

(vii) Uniformly Attracting in the Large: For all \( \rho, \eta \exists T(\eta, \rho) \)
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \eta \quad \forall \ t \geq t_0 + T. \]

(viii) Uniformly Asymptotically Stable in the Large (UASL)
\[ = \text{uniformly stable} + \]
\[ \text{uniformly bounded} + \]
\[ \text{uniformly attracting in the large.} \]
Exponential Asymptotic Stability

\[ \dot{x}(t) = f(x(t), t) \]

\[ x_0 \triangleq x(t_0) \]

Solution \( s(t; x_0, t_0) \)

**Definition:** (Malkin, 1935; Kalman and Bertram, 1960)

(i) **Exponentially Asymptotically Stable (EAS):**
\[ \forall \epsilon > 0 \ \exists \ \delta(\epsilon), \nu(\epsilon) \ \text{s.t.} \]
\[ \|x_0\| \leq \delta \implies \|s(t; x_0, t_0)\| \leq \epsilon e^{-\nu(t-t_0)} \]

(ii) **Exponentially Asymptotically Stable in the Large (EASL):**
\[ \forall \rho > 0 \ \exists \ \epsilon(\rho), \nu(\rho) \ \text{s.t.} \]
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \epsilon e^{-\nu(t-t_0)} \]

(iii) **Exponentially Stable (ES):**
\[ \forall \rho > 0 \ \exists \ \nu(\rho), \kappa(\rho) \ \text{s.t.} \]
\[ \|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \kappa \|x_0\| e^{-\nu(t-t_0)} \]

(iv) **Exponentially Stable in the Large (ESL):**
\[ \exists \nu, \kappa \ \text{s.t.} \]
\[ \|s(t; x_0, t_0)\| \leq \kappa \|x_0\| e^{-\nu(t-t_0)} \]
Rant about “uniform transients”