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Adaptive Systems with Closed–loop Reference Models: Composite control and observer feedback ⋆

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Abstract: A class of *Closed-loop Reference Models* (CRM) was shown in Gibson et al. (2013) to have improved transient performance. In this paper, we show that the introduction of CRM in *Combined direct and indirect Model Reference Adaptive Control* (CMRAC) leads to significant improvement in their transient response as well. We also show that CRM allow stable feedback of noise-free state estimates in CMRAC. Theoretical derivations are supported with numerical simulations.

Keywords: Adaptive control, Composite adaptive control, Robust adaptive control, Transient performance, Closed-loop reference model.

1. INTRODUCTION

Combined direct and indirect adaptive control, denoted as CMRAC, were examined in depth a few decades ago (see for example, Duarte and Narendra (1989); Slotine and Li (1989)). In these investigations, in addition to proving that these methods were stable, they also reported improved transient performance in simulations. We focus on this class of adaptive systems in this paper and introduce *Closed-loop Reference Models* (CRM)s into the picture. We show that the resulting adaptive systems, denoted as CMRAC-C, can be shown to have improved transients. For a class of plants where states are accessible, we show that CMRAC–C are stable, that together with an observer, denoted as CMRAC–CO, enable the feedback of noise-free state estimates while guaranteeing stability, and most importantly possess guaranteed transient properties similar to CRM control. These results are an extension of the results in Gibson et al. (2012, 2013).

The paper is organized as follows. We begin, in Section II, with CMRAC–C. In section III, the transient properties of CMRAC–C are investigated. In Section IV an observer feedback based CMRAC is introduced. Section V contains our concluding remarks.

2. STABILITY OF THE CMRAC–C

2.1 The Problem Statement and the CMRAC–C

In this section, we introduce the CRM and necessary definitions from Gibson et al. (2013). Consider the linear system dynamics with scalar input

$$\dot{x}(t) = A_p x(t) + bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, $A_p \in \mathbb{R}^{n\times n}$ is unknown and $b \in \mathbb{R}^n$ is known. Our goal is to design the control input such that $x(t)$ follows the reference model state $x_m(t) \in \mathbb{R}^n$ defined by the following dynamics

$$\dot{x}_m(t) = A_m x_m(t) + br(t) - L_m(x(t) - x_m(t)) \tag{2}$$

where $A_m \in \mathbb{R}^{n\times n}$ is Hurwtitz and $r(t) \in \mathbb{R}$ is a bounded possibly time varying reference command. $L_m \in \mathbb{R}^{n\times n}$ is denoted as the *Luenberger–gain*, and is chosen such that

$$\bar{A}_m \triangleq A_m + L_m \tag{3}$$

is Hurwitz. Equation (2) is referred to as a CRM, and when $L = 0$ the classical ORM is recovered.

Assumption 1. A parameter vector $\theta^* \in \mathbb{R}^n$ exists that satisfies the *matching condition*

$$A_m = A_p + b\theta^{*T}. \tag{4}$$

Assumption 2. A known θ^*_{max} exists such that $\|\theta^*\| \le \theta^*_{max}$.

The control input is chosen in the form

$$u(t) = \theta^T(t)x(t) + r(t) \tag{5}$$

where $\theta(t) \in \mathbb{R}^n$ is the adaptive control gain and signifies the direct component of the controller. We now present the indirect component. The identifier dynamics are given by

$$\dot{x}_i(t) = L_i(x_i(t) - x(t)) + (A_m - b\hat{\theta}^T(t))x(t) + bu(t) \tag{6}$$

where L_i is Hurwitz with $\hat{\theta}(t)$ signifying the indirect component of the controller. The error dynamics are now given by

$$\begin{aligned} \dot{e}_m(t) =&(A_m + L_m)e_m + b\tilde{\theta}^T(t)x, \quad e_m = x - x_m \\ \dot{e}_i(t) =&L_i e_i - b\bar{\theta}^T(t)x, \quad e_i = x_i - x \end{aligned} \tag{7}$$

where $\tilde{\theta}(t) = \theta(t) - \theta^*$ and $\bar{\theta}(t) = \hat{\theta}(t) - \theta^*$. The update laws for the two adaptive parameters is then

$$\begin{aligned} \dot{\theta} &= \mathrm{Proj}_\Gamma(\theta(t), -xe_m^T Pb, f) - \eta I_{n\times n}\epsilon_\theta \\ \dot{\hat{\theta}} &= \mathrm{Proj}_\Gamma(\hat{\theta}(t), xe_i^T P_i b, f) + \eta I_{n\times n}\epsilon_\theta \end{aligned} \tag{8}$$

where Proj_Γ is defined in (A.1), $\Gamma = \Gamma^T > 0$, $\eta > 0$, P and P_i are the solutions to

$$\bar{A}_m^T P + P\bar{A}_m = -I_{n\times n} \tag{9}$$

$$L_i^T P_i + P_i L_i = -I_{n\times n} \tag{10}$$

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and

$$f(\theta;\vartheta,\varepsilon)=\frac{\|\theta\|^2-\vartheta^2}{2\varepsilon\vartheta-\varepsilon^2} \tag{11}$$

where ϑ and ε are positive constants chosen as $\vartheta=\theta^*_{\max}$ and $\varepsilon>0$.

2.2 Preliminaries

All norms unless otherwise noted are the Euclidean–norm and the induced Euclidean–norm. The variable $t\in\mathbb{R}_+$ denotes time throughout and for a differentiable function $x(t)$, $\frac{d}{dt}x(t)$ is equivalent to $\dot{x}(t)$. Parameters explicit time dependence (t) is used upon introduction and then omitted thereafter except for emphasis. The other norms used in this work are the $\mathcal{L}_2$ and truncated $\mathcal{L}_2$ norm defined below. Given a vector $\nu\in\mathbb{R}^n$ and finite $p\in\mathbb{N}_{>0}$ $\|\nu(t)\|_{L_p}\triangleq\left(\int_0^\infty\|\nu(s)\|^p ds\right)^{1/p}$. The infinity norm is then defined as $\|\nu(t)\|_{L_\infty}\triangleq\sup\|\nu(t)\|$.

Definition 1. Given a Hurwtiz matrix $A_m\in\mathbb{R}^{n\times n}$

$$\begin{aligned}\sigma&\triangleq-\max_i\left(\mathrm{real}(\lambda_i(A_m))\right)\\ s&\triangleq-\min_i\left(\lambda_i\left(A_m+A_m^T\right)/2\right)\\ a&\triangleq\|A_m\|.\end{aligned} \tag{12}$$

For ease of exposition, throughout the paper, we choose L_m, L_i and Γ in (A.1) as follows:

$$L_m=-\ell I_{n\times n},\quad L_i\triangleq-(\sigma+\ell)I_{n\times n} \tag{13}$$

$$\Gamma\triangleq\gamma I_{n\times n}. \tag{14}$$

2.3 The Stability Result

Lemma 1. The constants σ and s are strictly positive and satisfy $s\geq\sigma>0$.

Lemma 2. With L_m chosen as in (13), A_m Hurwitz with constants σ and a as defined in (12), P in (9) satisfies

$$\text{(i)}\qquad \|P\|\leq\frac{m^2}{\sigma+2\ell} \tag{15}$$

$$\text{(ii)}\qquad \min_i\lambda_i(P)\geq\frac{1}{2(s+\ell)} \tag{16}$$

where $m=(1+4\varkappa)^{n-1}$ and $\varkappa\triangleq\frac{a}{\sigma}$, and P_i in (10) satisfies

$$P_i=\frac{1}{2(\sigma+\ell)}I_{n\times n} \tag{17}$$

Proof. See (Gibson et al., 2012, Lemma 2).

Definition 2. Using the design parameters of the convex function $f(\theta;\vartheta,\varepsilon)$ we introduce the following definitions

$$\begin{aligned}\theta_{\max}&\triangleq\vartheta+\varepsilon\text{ and}\\ \tilde{\theta}_{\max}&\triangleq2\vartheta+\varepsilon.\end{aligned} \tag{18}$$

Theorem 1. Let Assumptions 1 and 2 hold. Consider the overall CMRAC–C specified by (1), (2), (5), (6), (7) and (8). For any initial condition $e_m(0),e_i(0)\in\mathbb{R}^n$, and $\theta(0)$ and $\hat{\theta}(0)$ such that $\|\theta(0)\|\leq\theta_{\max}$ and $\|\hat{\theta}(0)\|\leq\theta_{\max}$, it can be shown that $e_m(t)$, $e_i(t)$, $\theta(t)$ and $\hat{\theta}(t)$ are uniformly bounded for all $t\geq0$ with $e_m(t)$ and $e_i(t)$ asymptotically converging to zero. The trajectories in the function

$$V=e_m^TPe_m+e_i^TP_ie_i+\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}+\bar{\theta}^T\Gamma^{-1}\bar{\theta} \tag{19}$$

converge exponentially to a set $\mathcal{E}$ as

$$\dot{V}\leq-\alpha_5V+\alpha_6 \tag{20}$$

where

$$\alpha_5\triangleq\frac{\sigma+2l}{m^2},\quad \alpha_6\triangleq\frac{2\alpha_5}{\gamma}\tilde{\theta}^2_{\max} \tag{21}$$

and

$$\mathcal{E}\triangleq\left\{(e_m,e_i,\tilde{\theta},\bar{\theta})\middle|\|e_m\|^2\leq\beta_4\tilde{\theta}^2_{\max},\|e_i\|^2\leq\beta_5\tilde{\theta}^2_{\max}\right.$$
$$\left.\|\tilde{\theta}\|\leq\tilde{\theta}_{\max},\ \|\bar{\theta}\|\leq\tilde{\theta}_{\max}\right\}$$

with

$$\beta_4\triangleq\frac{4(s+l)}{\gamma}\text{ and }\beta_5\triangleq\frac{4(\sigma+\ell)}{\gamma}. \tag{22}$$

Proof. see Appendix B.

3. TRANSIENT PROPERTIES OF CMRAC-C

In the following subsections we derive the transient properties of the CMRAC–C adaptive system, similar to what was done in Gibson et al. (2013). Two different subsections are presented, the first of which quantifies the Euclidean and the $\mathcal{L}_2$–norm of the tracking error e and the second subsection, where we define our metric for transient performance in terms of a truncated $\mathcal{L}_2$ norm of the rate of control effort.

Let

$$\rho=\frac{\gamma}{\sigma+\ell}. \tag{23}$$

The results in the following subsections are presented in terms of the two free design parameters ρ and ℓ, which is just a reparameterization of γ and ℓ. Then it is assumed that ρ is chosen independent of ℓ so that the product ΓP is of the same size while ℓ is being adjusted, where we note that

$$\|\Gamma\|\|P\|\leq\rho m^2. \tag{24}$$

This follows from the bound given in (15).

3.1 Bound on $e_m(t)$ and $e_i(t)$

Theorem 2. Let Assumptions 1 and 2 hold. Consider the overall CMRAC–C specified by (1), (2), (5), (6), (7) and (8). For any initial condition $e_m(0),e_i(0)\in\mathbb{R}^n$, and $\theta(0)$ and $\hat{\theta}(0)$ such that $\|\theta(0)\|\leq\theta_{\max}$ and $\|\hat{\theta}(0)\|\leq\theta_{\max}$.

$$\begin{aligned}\|e_m(t)\|^2\leq&\kappa_1\left(\|e_m(0)\|^2+\|e_i(0)\|^2\right)\exp\left(-\alpha_5t\right)\\ &+\frac{\kappa_2}{\rho}\tilde{\theta}^2_{\max}\end{aligned} \tag{25}$$

$$\|e_i(t)\|^2\leq\|e_m(t)\|^2 \tag{26}$$

$$\begin{aligned}\|e_m(t)\|^2_{L_2}\leq&\frac{1}{\sigma+\ell}\left(m^2\|e_m(0)\|^2+\|e_i(0)\|^2\right)\\ &+\frac{1}{\sigma+\ell}\left(\frac{1}{\rho}\|\tilde{\theta}(0)\|^2+\frac{1}{\rho}\|\bar{\theta}(0)\|^2\right)\end{aligned} \tag{27}$$

$$\|e_i(t)\|^2_{L_2}\leq\|e_m(t)\|^2_{L_2} \tag{28}$$

where κ_i, $i=1,2$ are independent of ρ and ℓ.

Proof. see Appendix C.

3.2 Bound on $\dot{u}(t)$

Definition 3. The following definitions will be useful when analyzing the transients of the CMRAC–C system:

$$\begin{aligned}\tau_3(\ell)\triangleq&\frac{2m^2}{\sigma+2\ell}\\ \tau_2\triangleq&\frac{2}{\sigma}\\ \delta_2(\ell,N)=&\exp\left(a_\theta N\tau_3(\ell)\right)-1\end{aligned} \tag{29}$$

where $a_\theta \triangleq a + \|b\|\tilde{\theta}_{\max}$. The time constant τ_3 will define the time constant for which we can upper bound the decay of the model following error and identification error. Similar to δ_1 in Gibson et al. (2013), δ_2 allows us to define the time scale separation condition for CMRAC–C which is defined in the following Lemma.

Lemma 3. Given an $N > 0$. An $\ell' > 0$ exists such that

(i) $\delta_2(\ell', N) < \delta$ where $0 < \delta \leq 1$.
(ii) $\tau_3(\ell') \leq \tau_2$.

Remark 1. Just as with the CRM adaptive system Gibson et al. (2013), N defines the number of time constants for which the error dynamics will decay, and thus in tern defines the ℓ' for which time scale separation holds.

Definition 4. The following three time intervals are used when exploring the transients of CMRAC–C

$$\begin{aligned} \mathbb{T}_1'' &= [0, N\tau_3) \\ \mathbb{T}_2'' &= [N\tau_3', T_1'') \\ \mathbb{T}_3'' &= [T_1'', \infty) \end{aligned} \tag{30}$$

where $T_1'' = \max\{N\tau_2, T(\epsilon, -\ell I_{n\times n})\}$, with $T(\epsilon, -\ell I_{n\times n})$ existing for any $\epsilon > 0$, this follows from the application of Barbalat Lemma to the adaptive system defined in Thereom 1 (identical to Corollary 2 in Gibson et al. (2013)).

Theorem 3. Let Assumptions 1–4 hold. Given arbitrary initial conditions in $x(0) \in \mathbb{R}^n$ and $\|\theta(0)\| \leq \theta_{\max}$, if $\ell \geq \ell'$ the derivative $\dot{u}$ satisfies the following two inequalities:

$$\begin{aligned} \sup_{t\in T_i''} |\dot{u}(t)| \leq &\left(\frac{m^2\gamma}{\sigma + 2\ell}\|b\|G_{e,i}''G_{x,i}'' + 8\eta\theta_{\max}^2\right) G_{x,i}'' \\ &+ \theta_{\max}\left(a_\theta G_{x,i}'' + r_0\right) + r_1 \end{aligned} \tag{31}$$

where

$$\begin{aligned} G_{x,1}'' \triangleq&(1+\delta_2)\|e(0)\| + \frac{\delta_2\|b\|}{a_\theta} r_0 \\ G_{e,1}'' \triangleq&\sqrt{\kappa_1}\left(\|e_m(0)\| + \|e_i(0)\|\right) + \sqrt{\frac{\kappa_2}{\rho}}\tilde{\theta}_{\max} \\ G_{x,2}'' \triangleq&\kappa_3\left(\|e_m(0)\| + \|e_i(0)\|\right) + (2+\kappa_4\ell)\sqrt{\frac{\kappa_2}{\rho}}\tilde{\theta}_{\max} \\ &+ \kappa_5 r_0 \\ G_{e,2}'' \triangleq&\sqrt{\kappa_1}\left(\|e_m(0)\| + \|e_i(0)\|\right)\epsilon_1 + \sqrt{\frac{\kappa_2}{\rho}}\tilde{\theta}_{\max} \\ G_{x,3}'' \triangleq&\kappa_6\left(\|e_m(0)\| + \|e_i(0)\|\right) + \epsilon \\ &+ (2+\kappa_4\ell)\sqrt{\frac{\kappa_2}{\rho}}\tilde{\theta}_{\max} + \kappa_5 r_0 \\ G_{e,3}'' \triangleq&\epsilon. \end{aligned} \tag{32}$$

with $\epsilon_1 = \exp(-N)$

Proof. The finite time stability result used in (Gibson et al., 2013, Appendix B) still holds for the MMRAC–C. Therefore $G_{x,1}''$ in (32) is identical to $G_{x,1}$ in (Gibson et al., 2013, (36)) with δ_2 replacing δ_1. The Lyapunov function in (19) has two additional terms in e_i and $\bar{\theta}$ as compared to the Lyapunov equation in (Gibson et al., 2013, (9)). Therefore, $G_{e,1}''$ now includes the initial conditions of the estimation error $e_i(0)$. $G_{x,2}''$ and $G_{e,2}''$ are similarly affected. Barbalat Lemma can be used for $G_{e,3}''$, and $G_{x,3}''$ follows from the same analysis in Gibson et al. (2013). The η terms arise from the righthand side the update law in (8).

The structure of the bounds in (32) is identical to that in (Gibson et al., 2013, (36)). Therefore this CMRAC–C will have the same"water–bed" effect as in direct CRM adaptive control case. This allows us to also conclude that an optimal selection of ρ and ℓ exists that minimizes the following cost function:

Theorem 4.

$$(\rho_{\text{opt}}, \ell_{\text{opt}}) = \underset{\substack{\rho>0 \\ \ell\geq\ell'}}{\arg\min}\|\dot{u}(\rho,\ell)\|_{L_2,\tau} \tag{33}$$

for any $0 < \tau < T_1''$.

4. CMRAC–CO

When measurement noise is present, it is often useful to use a state observer for feedback rather than the plant state. However, the use of such an observer in adaptive systems has proved to be quite difficult due to the inapplicability of the separation principle. In this section, we show how the CRM can be used to avoid this difficult for a class of plants. We denote the resulting adaptive system as CMRAC–CO.

We assume that the plant and reference model dynamics are given by Equations (1) and (2) with A_m and L_m satisfying Equations (4) and (3). The control input is now chosen as

$$u = \theta^T(t)x_o + r \tag{34}$$

and x_o is the state of the observer dynamics, given by

$$\dot{x}_o(t) = L_o(x_o(t) - x(t)) + (A_m - b\hat{\theta}^T(t))x_o(t) + bu(t). \tag{35}$$

Defining $e_m(t) = x(t) - x_m(t)$ and $e_o(t) = x_o - x(t)$, the error dynamics are now given by

$$\begin{aligned} \dot{e}_m(t) =&(A_m + L_m)e_m + b\tilde{\theta}^T(t)x_o + b\theta^* e_o \\ \dot{e}_o(t) =&(A_m + L_o - b\theta^*)e_o - b\bar{\theta}^T(t)x_o \end{aligned} \tag{36}$$

For ease of exposition we choose

$$L_m = L_o = -\ell I_{n\times n}. \tag{37}$$

The update laws for the adaptive parameters are then defined with the update law

$$\begin{aligned} \dot{\theta} &= \text{Proj}_\Gamma(\theta(t), -x_o e_m^T Pb, f) - \eta I_{n\times n}\epsilon_\theta \\ \dot{\hat{\theta}} &= \text{Proj}_\Gamma(\hat{\theta}(t), x_o e_o^T Pb, f) + \eta I_{n\times n}\epsilon_\theta \end{aligned} \tag{38}$$

with Γ chosen as in (14), $\eta > 0$, with P from (9) and $\epsilon_\theta = \theta - \hat{\theta}$.

Lemma 4. Let

$$\Delta(\ell) \triangleq \frac{4m^2\|b\|\theta_{\max}^*}{\sigma + 2\ell}. \tag{39}$$

Then, there exists an ℓ'' such that $0 < \Delta(\ell'') < 1$.

Theorem 5. Let Assumptions 1 and 2 hold with ℓ chosen such that $\ell \geq \ell''$. Consider the overall CMRAC–CO specified by (1), (2), (34), (35), (36) and (38). For any initial condition $e_m(0), e_o(0) \in \mathbb{R}^n$, and $\theta(0)$ and $\hat{\theta}(0)$ such that $\|\theta(0)\| \leq \theta_{\max}$ and $\|\hat{\theta}(0)\| \leq \theta_{\max}$, it can be shown that $e_m(t)$, $e_o(t)$, $\theta(t)$ and $\hat{\theta}(t)$ are uniformly bounded for all $t \geq 0$ and the trajectories in the function

$$V = e_m^T P e_m + e_o^T P_o e_o + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \bar{\theta}^T\Gamma^{-1}\bar{\theta} \tag{40}$$

converge exponentially to a set $\mathcal{E}$ as

$$\dot{V} \leq -\alpha_7 V + \alpha_8 \tag{41}$$

where

$$\begin{aligned} \alpha_7 \triangleq&\frac{(1-\Delta(\ell))(\sigma + 2\ell)}{m^2}, \\ \alpha_8 \triangleq&\frac{2(1-\Delta(\ell))(\sigma + 2\ell)}{\gamma m^2}\tilde{\theta}_{\max}^2 \end{aligned} \tag{42}$$

and

$$\mathcal{E} \triangleq \Big\{(e_m, e_o, \tilde{\theta}, \bar{\theta}) \Big| \|e_m\|^2 \leq \beta_6 \tilde{\theta}^2_{\max}, \|e_o\|^2 \leq \beta_6 \tilde{\theta}^2_{\max} \\ \|\tilde{\theta}\| \leq \tilde{\theta}_{\max},\ \|\bar{\theta}\| \leq \tilde{\theta}_{\max}\Big\}$$

with

$$\beta_6 \triangleq \frac{4(s+l)}{\gamma}. \tag{43}$$

Proof. see Appendix D.

4.1 Robustness of CMRAC–CO to Noise

As mentioned earlier, the benefits of the CMRAC–CO is the use of the observer state x_o rather than the actual plant state x. Suppose that the actual plant dynamics is modified from (1) as

$$\dot{x}_a(t) = A_p x_a(t) + bu(t), \qquad x(t) = x_a(t) + n(t) \tag{44}$$

where $n(t)$ represents measurement noise. For ease of exposition, we assume that $n(t)$ is a bounded, deterministic and time varying.

This leads to a set of modified error equations

$$\begin{aligned} \dot{e}_m(t) =& (A_m + L_m)e_m + b\tilde{\theta}^T(t)x_o + b\theta^* e_o + L_m n(t) \\ \dot{e}_o(t) =& (A_m + L_o - b\theta^*)e_o - b\bar{\theta}^T(t)x_o - L_o n(t) \end{aligned} \tag{45}$$

Theorem 6. Let Assumptions 1 and 2 hold with ℓ chosen such that $\ell \geq \ell''$. Consider the overall CMRAC–CO specified by (44), (2), (34), (35), (45) and (38). For any initial condition $e_m(0), e_o(0) \in \mathbb{R}^n$, and $\theta(0)$ and $\hat{\theta}(0)$ such that $\|\theta(0)\| \leq \theta_{\max}$ and $\|\hat{\theta}(0)\| \leq \theta_{\max}$, it can be shown that $e_m(t)$, $e_o(t)$, $\theta(t)$ and $\hat{\theta}(t)$ are uniformly bounded for all $t \geq 0$ and the trajectories in the function V from (40) converge exponentially to a set $\mathcal{E}$ as

$$\dot{V} \leq -\alpha_9 V + \alpha_{10} \tag{46}$$

where

$$\begin{aligned} \alpha_9 \triangleq& \frac{(1-\Delta(\ell))\,(\sigma + 2\ell)}{2m^2}, \\ \alpha_{10} \triangleq& \frac{(1-\Delta(\ell))\,(\sigma+2\ell)}{\gamma m^2}\tilde{\theta}^2_{\max} \\ &+ \frac{16}{(1-\Delta(\ell))^2}\left(\frac{m^2}{\sigma+2\ell}\right)^2 \|n(t)\|^2 \end{aligned} \tag{47}$$

and

$$\mathcal{E} \triangleq \Big\{(e_m, e_o, \tilde{\theta}, \bar{\theta}) \Big| \|e_m\|^2 \leq \beta_6\tilde{\theta}^2_{\max} + \beta_7\|n(t)\|^2, \\ \|e_o\|^2 \leq \beta_6\tilde{\theta}^2_{\max} + \beta_7\|n(t)\|^2, \\ \|\tilde{\theta}\| \leq \tilde{\theta}_{\max},\ \|\bar{\theta}\| \leq \tilde{\theta}_{\max}\Big\}$$

with β_6 defined in (43) and β_7 defined as

$$\beta_7 \triangleq \frac{64m^2 s}{\sigma(1-\Delta(\ell))^3} \tag{48}$$

Proof. see Appendix E

4.2 Simulation Study

For this study CMRAC-CO is compared to CMRAC in the presence of noise. The plant dynamics under study are the linear short-period dynamics of an F-16 Aircraft derived from (Stevens and Lewis, 2003, Table 3.4-3, Example 5.5-3 Appendix A). For this example the states of the plant are the angle of attack α [rad], and pitch rate q [rad/s]. The control input u is the elevator deflection in [deg]. We note that the angles are mixed between radians and degrees, but that is the convention used in Stevens and Lewis (2003). The reference model Jacobian is taken directly from the text and the plant Jacobian is modified so that the open–loop plant is unstable. The CMRAC controller is defined by (44), (2), (5), (6), (7) and (8) where $L_m = 0$, denoting an open–loop reference model. The CMRAC-CO is defined by (44), (2), (34), (35), (45) and (38). In the following simulations the aircraft is given a square wave reference input to the elevators and the pitch rate is initialized at 0.3 [rad/s]. The noise affects both the pitch rate measurement and angle of attack measurement independently and is generated from a Gausian distribution with standard deviation 0.1, deterministically sampled using a fixed seed at 100 Hz. All plant parameters and control parameters are given in Table 1. We also note that for the linear short period dynamics of an aircraft the angle of attack mimics the behavior of the pitch rate, and thus the angle of attack trajectories are not included as they do not give any further insight into the performance of the adaptive systems.

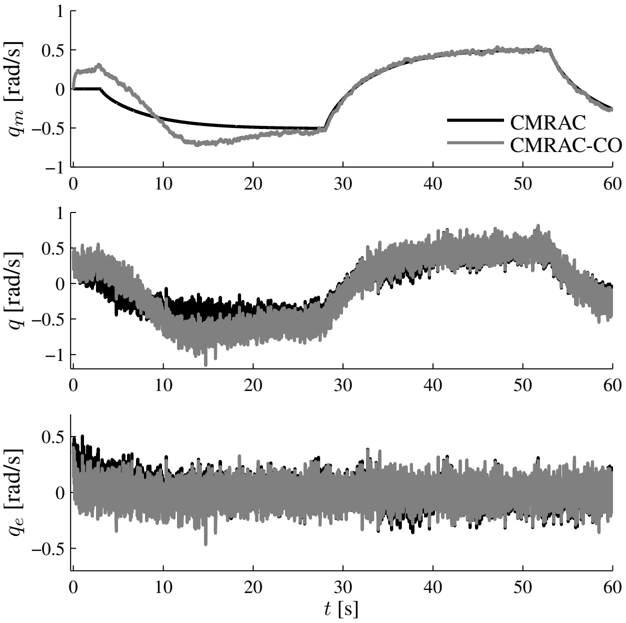

Fig. 1. Reference model, state and error for the simulation study.

The simulation results are contained in Figures 1 and 2. Figure 1 contains the pitch rate reference, the measured pitch rate of the plant and the pitch rate error, denoted as q_m, q, q_e respectively.

Table 1. Test case free design parameters

Paramater	Value
$x(0)^T$	$\begin{bmatrix}0 & 0.3\end{bmatrix}$
A_p	$\begin{bmatrix}-1.0 & 0.9 \\ 0.8 & -1.1\end{bmatrix}$
A_m	$\begin{bmatrix}-1.0 & 0.9 \\ 1 & 0.6\end{bmatrix}$
b^T	$\begin{bmatrix}0 & -0.2\end{bmatrix}$
θ^{*T}	$\begin{bmatrix}1 & 3\end{bmatrix}$
L_m	$-10I^{n\times n}$
$L_{i,o}$	$-10I^{n\times n}$
η	1
γ	$100I^{n\times n}$

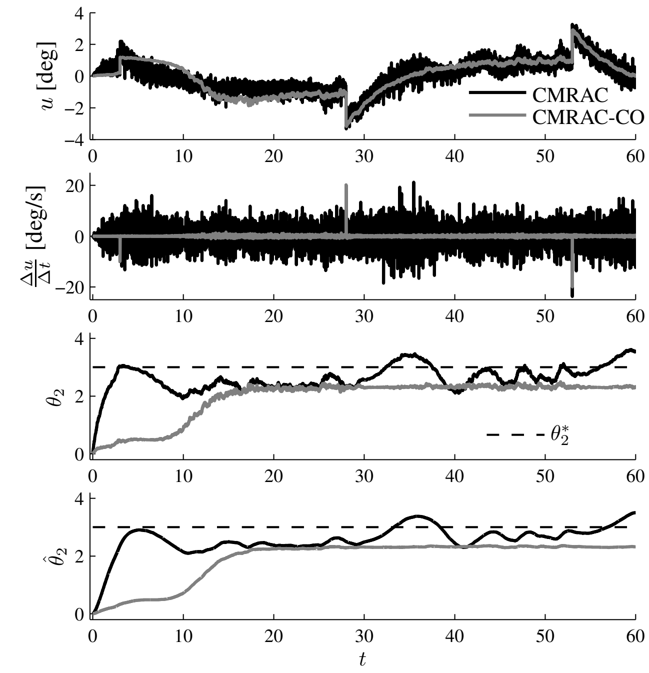

Fig. 2. Control input, discrete rate of control input and the adaptive parameters for the simulation study.

Figure 2 contains the the control input, the discrete difference of control input, the second element in $\theta(t)$ and the second element in $\hat{\theta}(t)$. The first thing to note is the difference in the behavior of the reference signal q_m for the CMRAC and CMRAC-CO systems. CMRAC uses an open–loop reference model, and therefore the trajectory of q_m is independent from the plant and results in a low pass filtered step response. The reference q_m for the CMRAC-CO however, immediately increases to approximately 0.3 [rad/s], the initial condition of q, and then asymptotically converges to its open-loop counterpart. This is the major trade off that one must realize when using CRM adaptive systems. The reference models are no longer a-priori known for a given r. The affect of filtering the regressor and using observer state feedback is readily visible when comparing the control input and $\Delta u/\Delta t$ for the two controllers. Much less of the measurement noise is passed onto the control input in the CMRAC-CO system. The large jumps in $\Delta u/\Delta t$ are from the steps in the square wave command r. For the design parameters in this system the CMRAC adaptive parameters are on the verge of departing to their projection limits. The full departure was observed when either the noise level was increased or the tuning gain Γ was increased.

5. CONCLUSIONS

As discussed in the Introduction, combining indirect and direct adaptive control has always been observed to produce desirable transient response in adaptive control. While the above analysis does not directly support the observed transient improvements with CMRAC, we provide a few speculations below: The free design parameter L_i in the identifier is typically chosen to have eigenvalues faster than the plant that is being controlled. Therefore the identification model following error e_i converges rapidly and $\hat{\theta}(t)$ will have smooth transients. It can be argued that the desirable transient properties of the identifier pass on to the direct component through the tuning law, and in particular ϵ_θ.

The CMRAC–C differs from classical CMRAC only due to the feedback gain L_m in the reference model. Given the contributions of Gibson et al. (2013) which show that the CRM can result in satisfactory transients without the indirect component raises the question if the added complexity of a CMRAC–C is justified. One answer to this question is in the form of the CMRAC–CO, where it is shown that one can design stable observer–based feedback in a CMRAC, allowing noise-free estimation and control.

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Appendix A. PROJECTION OPERATOR

The Γ–*Projection Operator* for two vectors $\theta, y \in \mathbb{R}^k$, a convex function $f(\theta) \in \mathbb{R}$ and with symmetric positive definite tuning gain $\Gamma \in \mathbb{R}^{k\times k}$ is defined as

$$\mathrm{Proj}_\Gamma(\theta, y, f) = \begin{cases} \Gamma y - \Gamma \dfrac{\nabla f(\theta)(\nabla f(\theta))^T}{(\nabla f(\theta))^T \Gamma \nabla f(\theta)} \Gamma y f(\theta) \\ \qquad \text{if } f(\theta) > 0 \wedge y^T \Gamma \nabla f(\theta) > 0 \\ \Gamma y \qquad \text{otherwise} \end{cases} \tag{A.1}$$

where $\nabla f(\theta) = \left(\frac{\partial f(\theta)}{\partial \theta_1} \ \cdots \ \frac{\partial f(\theta)}{\partial \theta_k} \right)^T$. The projection operator was first introduced in Pomet and Praly (1992) with extensions in Ioannou and Sun (1996) and for a detailed analysis of Γ–projection see Lavretsky and Gibson (2011).

Appendix B. PROOF OF THEOREM 1

Proof. Taking the time derivative of V in (19) results in

$$\dot{V} \leq -\|e_m\|^2 - \|e_i\|^2 - 2\frac{\eta}{\gamma}\epsilon_\theta^2. \tag{B.1}$$

Substitution of V in (19) results in

$$\dot{V} \leq -\alpha_5 V + \alpha_6 \tag{B.2}$$

where α_5 and α_6 are defined in (21). Using the bound in Lemma 2–(ii) we have that

$$e_m^T P_m e_m \geq \frac{1}{2(s+\ell)}\|e_m\|^2 \text{ and } e_i^T P_i e_i \geq \frac{1}{2(\sigma+\ell)}\|e_i\|^2$$

then we can conclude that $\lim_{t\to\infty}\|e_m(t)\|^2 \leq \beta_4\tilde{\theta}^2_{\max}$ and $\lim_{t\to\infty}\|e_i(t)\|^2 \leq \beta_5\tilde{\theta}^2_{\max}$ where β_4 and β_5 are defined in (48). The boundedness of $\theta(t)$ and $\hat{\theta}(t)$ follows from the use of a projection algorithm. The asymptotic limit to zero comes from the application of Barbalat Lemma.

Appendix C. PROOF OF THEOREM 2

The bounds in (25) and (26) follow from the application of Gronwall–Bellman to the result in (20) with the lower bound for $\min \lambda_i(P)$ in (16) and the change of parameters from (23) being used.

Beginning with

$$\begin{aligned}\|e_m(t)\|^2_{L_2} \leq & \int_0^\infty -\dot{V}(e(t),\tilde{\theta}(t)) \leq V(e(0),\tilde{\theta}(0))\\ \leq & \frac{m^2}{\sigma+2\ell}\|e_m(0)\|^2 + \frac{1}{2(\sigma+\ell)}\|e_i(0)\|^2 \\ & + \frac{2}{\gamma}\|\tilde{\theta}(0)\|^2,\end{aligned} \tag{C.1}$$

using the definitions of ρ from (23), the fact that $\frac{1}{\sigma+2\ell} \leq \frac{1}{\sigma+\ell}$ the bound in (27) holds. This same approach can be used to obtain the bound in (28).

Appendix D. PROOF OF THEOREM 5

Proof. Taking the time derivative of V in (40) results in

$$\dot{V} \leq -(1-\Delta(\ell))\left(\|e_m\|^2 + \|e_o\|^2\right) - 2\frac{\eta}{\gamma}\epsilon^2_\theta. \tag{D.1}$$

where $\Delta(l)$ is defined in (39). Substitution of V in (40) results in

$$\dot{V} \leq -\alpha_7 V + \alpha_8 \tag{D.2}$$

where α_7 and α_8 are defined in (42). Using the bound in Lemma 2–(ii) we have that

$$e_m^T P e_m \geq \frac{1}{2(s+\ell)}\|e_m\|^2 \text{ and } e_o^T P e_o \geq \frac{1}{2(s+\ell)}\|e_o\|^2$$

then we can conclude that $\lim_{t\to\infty}\|e_m(t)\|^2 \leq \beta_6\tilde{\theta}^2_{\max}$ and $\lim_{t\to\infty}\|e_o(t)\|^2 \leq \beta_6\tilde{\theta}^2_{\max}$ where β_6 is defined in (43). The boundedness of $\theta(t)$ and $\hat{\theta}(t)$ follows from the use of a projection algorithm.

Appendix E. PROOF OF THEOREM 6

Proof. Taking the time derivative of V in (40) results in

$$\begin{aligned}\dot{V} \leq & -(1-\Delta(\ell))\left(\|e_m\|^2 + \|e_o\|^2\right) - 2\frac{\eta}{\gamma}\epsilon^2_\theta \\ & + 2\|P\|\|n(t)\|\|e_m(t)\| + 2\|P\|\|n(t)\|\|e_o(t)\|\end{aligned}. \tag{E.1}$$

completing the square in $\|e_m\|\|n\|$ and $\|e_o\|\|n\|$

$$\begin{aligned}\dot{V} \leq & -\frac{(1-\Delta(\ell))}{2}\left(\|e_m\|^2 + \|e_o\|^2\right) - 2\frac{\eta}{\gamma}\epsilon^2_\theta \\ & - \frac{(1-\Delta(\ell))}{2}\left(\|e_m\| - \frac{4}{(1-\Delta(\ell))}\|P\|\|n(t)\|\right)^2 \\ & - \frac{(1-\Delta(\ell))}{2}\left(\|e_o\| - \frac{4}{(1-\Delta(\ell))}\|P\|\|n(t)\|\right)^2 \\ & + \frac{16}{(1-\Delta(\ell))^2}\|P\|^2\|n(t)\|^2.\end{aligned} \tag{E.2}$$

Neglecting the negative terms in lines 2 and 3 from the equation above and substitution of the norm for P we have that

$$\begin{aligned}\dot{V} \leq & -\frac{(1-\Delta(\ell))}{2}\left(\|e_m\|^2 + \|e_o\|^2\right) - 2\frac{\eta}{\gamma}\epsilon^2_\theta \\ & + \frac{16}{(1-\Delta(\ell))^2}\|P\|^2\|n(t)\|^2.\end{aligned} \tag{E.3}$$

which in terms of V is identical to

$$\begin{aligned}\dot{V} \leq & -\frac{(1-\Delta(\ell))(\sigma+2\ell)}{2m^2}V + \frac{(1-\Delta(\ell))(\sigma+2\ell)}{\gamma m^2}\tilde{\theta}^2_{\max} \\ & + \frac{16}{(1-\Delta(\ell))^2}\left(\frac{m^2}{\sigma+2\ell}\right)^2\|n(t)\|^2.\end{aligned} \tag{E.4}$$

$$\dot{V} \leq -\alpha_9 V + \alpha_{10} \tag{E.5}$$

where α_9 and α_{10} are defined in (47). Using the bound in Lemma 2–(ii) we can conclude that

$$\lim_{t\to\infty}\|e_m(t)\|^2 \leq \beta_6\tilde{\theta}^2_{\max} + \beta_7\|n(t)\|^2$$

and

$$\lim_{t\to\infty}\|e_o(t)\|^2 \leq \beta_6\tilde{\theta}^2_{\max} + \beta_7\|n(t)\|^2$$

where β_7 is defined in (48). The boundedness of $\theta(t)$ and $\hat{\theta}(t)$ follows from the use of a projection algorithm.