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# Adaptive Control and the Definition of Exponential Stability

Travis E. Gibson and Anuradha M. Annaswamy

***Abstract*—Several definitions of exponential stability are revisited so as to address some possible confusion in the adaptive control literature when terms like exponentially convergent, exponentially asymptotically stable, or exponentially stable are used. It is also shown that in general the direct adaptive control problem can never be exponentially stable in the large and can at best be uniformly asymptotically stable in the large.**

## I. Introduction

As the analysis of parameter convergence in adaptive control wound through the 60's 70's and 80's [1, 3–5, 12, 13, 17–19, 22] the persistence of excitation condition was moved from the regressor vector to the reference input, and the stability was shown to be exponential under checkable conditions on an external signal. The technical bridge for this jump came via the stability analysis of a class of linear time varying systems in [2]. There has been a misunderstanding, however, on what parameters should be independent when determining whether a system is exponentially stable.

Persistence of excitation of the reference input, also referred to as sufficient richness, does not imply persistence of excitation of the regressor vector. The authors of [4, 5] are careful in proving that richness of the reference input only implies exponential convergence. The careful wording of *convergence* however is changed to exponential *stability* countless times elsewhere in the literature. It is then inappropriately concluded that uniform asymptotic stability in the large is equivalent to exponential stability in the large for adaptive systems.

This note is intended to be a cautionary tale and complements the works of [21] and [14] in carefully defining persistence of excitation and a weaker version that is not uniform in initial conditions. Where as [21] and [14] focus on the various stability results when two different kinds of persistence of excitation are studied, we illustrate why in general adaptive systems can never satisfy the original strong version of persistence of excitation. Along that ideology we pick up where [19] left off and in so doing hope to clarify the true stability properties of adaptive systems. An interesting perspective on the similarity and differences between exponential and asymptotic stability is given in [6], but we are interested in a simpler treatment.

The main contribution of this work is a detailed account of where caution should be exercised when taking the persistence of excitation condition from the reference model state to the regressor, and establish that adaptive systems arising in the control of unknown linear plants are at best uniformly asymptotically stable in the large. This paper is organized as follows: Section II contains stability definitions, Section III contains the definition of persistence of excitation, Section IV contains a detailed analysis of the convergence properties of two sets of dynamics occurring in adaptive systems. Section V closes with a discussion.

T. E. Gibson is with Harvard Medical School and the Channing Division of Network Medicine, Department of Medicine, Brigham and Women's Hospital, Boston MA, 02115, email: (tgibson@mit.edu)

A. M. Annaswamy is with the Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA, 02139.

## II. Stability

Consider a dynamical system defined by the following relations

$$\begin{aligned} x(t_0) &= x_0 \\ \dot{x}(t) &= f(x(t), t) \end{aligned}$$

where $t \in [t_0, \infty)$ is time and $x \in \mathbb{R}^n$ denotes the state vector. We are interested in systems with equilibrium at $x = 0$, so that $f(0, t) = 0$ for all $t$. The solution to the differential equation above for $t \geq t_0$ is a transition function $s(t; x_0, t_0)$ such that $\dot{s}(t; x_0, t_0) = f(s(t; x_0, t_0), t)$ and $s(t_0; x_0, t_0) = x_0$. Various definitions of stability now follow [7, 11, 16].

**Definition 1** (Stability and Asymptotic Stability)**.** Let $t_0 \geq 0$, the equilibrium is

(i) *Stable*, if for all $\epsilon > 0$ there exists a $\delta(\epsilon, t_0) > 0$ such that $\|x_0\| \leq \delta$ implies $\|s(t; x_0, t_0)\| \leq \epsilon$ for all $t \geq t_0$.
(ii) *Attracting*, if there exists a $\rho(t_0) > 0$ such that for all $\eta > 0$ there exists an attraction time $T(\eta, x_0, t_0)$ such that $\|x_0\| \leq \rho$ implies $\|s(t; x_0, t_0)\| \leq \eta$ for all $t \geq t_0 + T$.
(iii) *Asymptotically Stable*, if it is stable and attracting.
(iv) *Uniformly Stable* if the $\delta$ in (i) is uniform in $t_0$ and $x_0$, thus taking the form $\delta(\epsilon)$.
(v) *Uniformly Attracting*, if it is attracting where the $\rho$ and $T$ do not depend on $t_0$ or $x_0$ and thus the attracting time take the form $T(\eta, \rho)$.
(vi) *Uniformly Asymptotically Stable*, (UAS) if it is uniformly stable and uniformly attracting.
(vii) *Uniformly Bounded* if for all $r > 0$ there exists a $B(r)$ such that $\|x_0\| \leq r$ implies that $\|s(t; x_0, t_0)\| \leq B$ for all $t \geq t_0$.
(viii) *Uniformly Attracting in the Large* if for all $\rho > 0$ and $\eta > 0$ there exists a $T(\eta, \rho)$ such that $\|x_0\| \leq \rho$ implies $\|s(t; x_0, t_0)\| \leq \eta$ for all $t \geq t_0 + T$.
(ix) *Uniformly Asymptotically Stable in the Large* (UASL) if it is uniformly stable, uniformly bounded, and uniformly attracting in the large.

The definitions regarding stability and asymptotic stability are universally accepted and repeated consistently in the

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literature. The definition of exponential stability however has not had the same consistent treatment. This is most likely due to the fact that for linear systems, they all become equivalent. The first formal definition of exponential stability is credited by [16] to have first appeared in [15] and repeated here in: the state trajectory $s(t)$ is exponentially asymptotically stable if there exists $\nu > 0$ and for all $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that $\|x_0\| \leq \delta$ implies that $\|s(t; x_0, t_0)\| \leq \epsilon\, \mathbf{e}^{-\nu(t-t_0)}$. The term exponential asymptotic stability is no longer used in modern control text and the shortened phrase exponential stability is used instead. There are however several different ways to classifiy systems that portray exponential convergence. We now give four different definitions of exponential stability, beginning with a slightly modified version of the exponential asymptotic stability definition given in [16].

**Definition 2** (Exponential Stability)**.** Let $t_0 \geq 0$, the equilibrium is

(i) *Exponentially Asymptotically Stable* (EAS) if for all $\epsilon > 0$ there exists a $\delta(\epsilon), \nu(\epsilon) > 0$ such that $\|x_0\| \leq \delta$ implies that $\|s(t; x_0, t_0)\| \leq \epsilon\, \mathbf{e}^{-\nu(t-t_0)}$

(ii) *Exponentially Asymptotically Stable in the Large* (EASL) if for all $\rho > 0$ there exists an $\epsilon(\rho), \nu(\rho) > 0$ such that $\|x_0\| \leq \rho$ implies that $\|s(t; x_0, t_0)\| \leq \epsilon\, \mathbf{e}^{-\nu(t-t_0)}$

(iii) *Exponentially Stable* (ES) if for every $\rho > 0$ there exists $\nu(\rho) > 0$ and $\kappa(\rho) > 0$ such that $\|x_0\| \leq \rho$ implies $\|s(t; x_0, t_0)\| \leq \kappa \|x_0\| \mathbf{e}^{-\nu(t-t_0)}$

(iv) *Exponentially Stable in the Large* (ESL) if there exists $\nu > 0$ and $\kappa > 0$ such that $\|s(t; x_0, t_0)\| \leq \kappa \|x_0\| \mathbf{e}^{-\nu(t-t_0)}$ for all $x_0$.

The definition of EAS is repeated here only for historical context. The definition is then extended analogously to UASL with EASL, where the size of the neighborhoods for the initial condition can be arbitrarily large. It is clear from the definition that EASL implies UASL, let $T(\rho, \eta) = \frac{1}{\nu(\rho)} \log\left(\frac{\epsilon(\rho)}{\eta}\right)$. The same definition can be used to show that EAS implies UAS. A slightly stronger version of EASL is that of ES. The exponential stability definition is motivated by the analysis of linear differential equations, where it is easy to explicitly extract $x_0$ from $\epsilon$ in the definition. Finally, ESL was introduced which is stronger than ES due to the fact that $\kappa$ and most importantly $\nu$ hold globally and uniformly. Note that for linear systems, i.e. $\dot{x} = A(t)x$, UAS implies ESL [11, Theorem 3: (C) & (D)]. Thus, for linear systems all of the definitions are equivalent. The relationship between these definitions of stability are illustrated in the following implication diagram.

$$
\begin{tikzcd}
\text{ESL} \arrow[r, Rightarrow] & \text{ES} \arrow[r, Rightarrow] & \text{EASL} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{EAS} \arrow[d, Rightarrow] \\
 & & \text{UASL} \arrow[r, Rightarrow] & \text{UAS} \arrow[lllu, Rightarrow, bend left, "+\text{ Linear}"]
\end{tikzcd}
$$

**Theorem 1.** *Consider the scalar dynamics*

$$\dot{x}(t) = a(t)x(t).$$

*If there exists $T, \alpha_1, \alpha_2 > 0$ such that $\int_t^{t+T} a(\tau)d\tau \leq -\alpha_1$ and $\infty < a(t) \leq \alpha_2$ for all $t \geq t_0$, then the scalar dynamics just mentioned are ESL with $\kappa = \mathbf{e}^{\alpha_1 + \alpha_2 T}$ and $\nu = \alpha_1 / T$.*

**Corollary.** *The scalar dynamics $\dot{x}(t) = -y^2(t)x(t)$ are ESL if there exists positive constants $\alpha_1$ and $T$ such that $\int_t^{t+T} y^2(\tau)\, d\tau \geq \alpha_1$, and $y$ is bounded for all $t \geq t_0$.*

A proof of the above theorem is given in the Appendix. The previous theorem illustrates that if $a(t)$ is on average always negative over some sliding window in time of size $T$, then the system is exponentially stable in the large. We now give an example of a system that is exponentially convergent, yet is neither ESL or ES. How well the dynamics satisfy the sufficient conditions of Theorem 1 are then analyzed.

**Example 1.** Consider the differential equation

$$\dot{x}(t) = -|x_0|x(t).$$

It follows that $s(t; x_0, t_0) = \mathbf{e}^{-|x_0|(t-t_0)}x_0$ is a solution to the differential equation of interest. While the dynamics are exponentially decreasing for all nonzero initial conditions, the exponent can not be lower bounded for any neighborhood of the origin. Thus, the dynamics are not ESL or ES. The sufficient conditions of Theorem 1 are now investigated. Let $\alpha_1 = T|x_0|$ and $\alpha_2 = 0$, therefore

$$|s(t; x_0, t_0)| \leq \mathbf{e}^{T|x_0|}\mathbf{e}^{-|x_0|(t-t_0)}|x_0|.$$

Taking the limit as $T$ tends to zero in the above inequality we recover the least upper bound of $|s(t; x_0, t_0)| \leq \mathbf{e}^{-|x_0|(t-t_0)}|x_0|$. Even though the sufficient conditions of Theorem 1 are not satisfied, as $\alpha_1$ tends to zero as $x_0$ tends to zero, this illustrates that the theorem is not overly conservative.

The example just presented seems to be completely unfeasible. In the strict equality we agree, there are not many systems where the decay rate is strictly proportional to the initial condition. However, as will be shown later, in adaptive control the scenario is often encountered where the convergence rate of a function is bounded by the initial conditions of the system.

## III. Persistence of Excitation

**Definition 3** (Persistence of Excitation)**.** Let $\omega \in [t_0, \infty) \to \mathbb{R}^p$ be a time varying parameter with initial condition defined as $\omega_0 = \omega(t_0)$, then the parameterized function of time $y(t, \omega) : [t_0, \infty) \times \mathbb{R}^p \to \mathbb{R}^m$ is

(i) *Persistently Exciting* (PE) if there exists $T > 0$ and $\alpha > 0$ such that

$$\int_t^{t+T} y(\tau, \omega)y^{\mathsf{T}}(\tau, \omega)d\tau \geq \alpha I$$

for all $t \geq t_0$ and $\omega_0 \in \mathbb{R}^p$, and we denote this as $y(t, \omega) \in$ PE.

(ii) *weakly Persistently Exciting* ($\text{PE}^*(\omega,\Omega)$) if there exists a compact set $\Omega \subset \mathbb{R}^p$, $T(\Omega) > 0, \alpha(\Omega)$ such that

$$\int_t^{t+T} y(\tau,\omega)y^{\mathsf{T}}(\tau,\omega)d\tau \geq \alpha I$$

for all $\omega_0 \in \Omega$ and $t \geq t_0$, and we denote this as $y(t,\omega) \in \text{PE}^*(\omega,\Omega)$.

## IV. Asymptotic and Exponential Stability in Adaptive Control

We now present two adaptive systems, first the algebraic identification problem and then the standard adaptive control problem.

*Identification in Simple Algebraic Systems* [20]

Let $u : [t_0,\infty) \to \mathbb{R}^n$ be the input and $y : [t_0,\infty) \to \mathbb{R}$ be the output of the following algebraic system of equations

$$y(t) = u^{\mathsf{T}}(t)\theta$$

where $\theta \in \mathbb{R}^n$ is an unknown parameter. If we assume that $u$ is known and $y$ is measurable, then an estimate of the unknown parameter $\hat{\theta} : [,t_0,\infty) \to \mathbb{R}^n$ can be used in constructing an adaptive observer

$$\hat{y}(t) = u^{\mathsf{T}}(t)\hat{\theta}(t)$$

where the update for the estimate of the uncertain parameter is defined as

$$\dot{\hat{\theta}}(t) = -u(t)\left(\hat{y}(t) - y(t)\right).$$

Denoting the parameter error as $\phi(t) = \hat{\theta}(t) - \theta$ the parameter error evolves as

$$\dot{\phi}(t) = -u(t)u^{\mathsf{T}}(t)\phi(t). \tag{1}$$

**Theorem 2.** *If $u(t)$ is PE and either 1) there exists $\beta > 0$ such that*

$$\int_t^{t+T} u(\tau)u^{\mathsf{T}}(\tau)d\tau \leq \beta I$$

*or 2) there exists a $u_{\max} > 0$ such that $\|u(t)\| \leq u_{\max}$, then $\phi$ in equation* (1) *is ESL.*

The proof is given in two flavors the first follows that of [1] and the second follows that of [20], and then the two methods are compared.

*Proof of the theorem following Anderson* [1, proof of Theorem 1]*:* Note that the existence of a $u_{\max}$ and a $T$ that is finite implies the existence of $\beta$. The existence of $T$, $\alpha$, and $\beta$ such that $\alpha I \leq \int_t^{t+T} u(\tau)u^{\mathsf{T}}(\tau)d\tau \leq \beta I$ is equivalent to the following system being *uniformly completely observable* $\Sigma_1 :\ \dot{x}_1 = 0_{n\times n}x_1,\ y_1 = u^{\mathsf{T}}(t)x_1$ [10, Definition (5.23) dual of (5.13)]. If $\Sigma_1$ is uniformly completely observable then if follows that $\Sigma_2 : \dot{x}_2 = -u(t)u^{\mathsf{T}}(t)x_2,\ y_2 = u^{\mathsf{T}}(t)x_2$ is uniformly completely observable as well [2, Dual of Theorem 4]. Therefore, there exists $\alpha_2$ and $\beta_2$ such that

$$\alpha_2 I \leq \int_t^{t+T} \Phi_2^{\mathsf{T}}(\tau,t)u(\tau)u^{\mathsf{T}}(\tau)\Phi_2(\tau,t)d\tau \leq \beta_2 I \tag{2}$$

where $\Phi_2(t,t_0)$ is the state transition matrix for $\Sigma_2$. Note that the upper bound $\beta$ is needed to ensure that $\Phi_2(\tau,t)$ is not singular, $\det\Phi_2(t,t_0) = \exp\left[-\int_{t_0}^{t}\text{trace}(u(\tau)u^{\mathsf{T}}(\tau))\,d\tau\right]$.

Let $V(\phi,t) = \frac{1}{2}\phi^{\mathsf{T}}(t)\phi(t)$ and note that $\Sigma_2$ and (1) have the same state transition matrix. Thus $\phi(t;t_0) = \Phi_2(t,t_0)\phi(t_0)$. Differentiating $V$ along the system trajectories in (1) we have $\dot{V}(\phi,t;t_0) = -\phi^{\mathsf{T}}(t_0)\Phi_2^{\mathsf{T}}(t,t_0)u(t)u^{\mathsf{T}}(t)\Phi_2(t,t_0)\phi(t_0)$. Using the bound in (2) and integrating as $\int_t^{t+T}\dot{V}(\phi,\tau;t)d\tau$, it follows that $V(t+T) - V(t) \leq -2\alpha_2 V(t)$. Thus $V(t+T) \leq (1 - 2\alpha_2)V(t)$ and therefore the system is UASL and due to linearity it follows that the systems is ESL. ∎

*Proof of the theorem following Narendra and Annaswamy* [20, proof of Theorem 2.16]*:* First we note that $u(t)$ being PE is equivalent to

$$\int_t^{t+T} |u^{\mathsf{T}}(\tau)w|^2 d\tau \geq \alpha$$

holding for any fixed unitary vector $w$. Let $\tilde{u}(t) \triangleq \frac{u(t)}{u_{\max}}$, then it follows that

$$\begin{aligned}\int_t^{t+T} |u^{\mathsf{T}}(\tau)w|^2 d\tau &= u_{\max}^2 \int_t^{t+T} |\tilde{u}^{\mathsf{T}}(\tau)w|^2 d\tau \\ &\leq u_{\max}^2 \int_t^{t+T} |\tilde{u}^{\mathsf{T}}(\tau)w| d\tau\end{aligned}$$

where the second line of the above inequality follows due to the fact that $\|\tilde{u}\| \leq 1$ and thus $|\tilde{u}^{\mathsf{T}}(\tau)w|^2 \leq |\tilde{u}^{\mathsf{T}}(\tau)w|$. Therefore, $u$ being PE and bounded implies that

$$\frac{\alpha}{u_{\max}} \leq \int_t^{t+T} |u^{\mathsf{T}}(\tau)w| d\tau. \tag{3}$$

The above bound will be called upon shortly. Moving forward with the proof, consider the Lyapunov candidate $V(\phi,t) = \frac{1}{2}\phi^{\mathsf{T}}(t)\phi(t)$. Then differentiating along the system directions it follows that $\dot{V}(\phi,t) = -\phi^{\mathsf{T}}(t)u(t)u^{\mathsf{T}}(t)\phi(t)$. Integrating $\dot{V}$ and using the Cauchy Schwartz inequality it follows

$$\begin{aligned}-\int_t^{t+T} \dot{V}(\phi,\tau)d\tau &= \int_t^{t+T} |u^{\mathsf{T}}(\tau)\phi(\tau)|^2 d\tau \\ &\geq \frac{1}{T}\left(\int_t^{t+T} |u^{\mathsf{T}}(\tau)\phi(\tau)| d\tau\right)^2.\end{aligned}$$

The above inequality can equivalently be written as

$$\sqrt{T(V(t) - V(t+T))} \geq \int_t^{t+T} |u^{\mathsf{T}}(\tau)\phi(\tau)| d\tau. \tag{4}$$

Using the reverse triangle inequality, the righthand side of the inequality in (4) can be bounded as

$$\begin{aligned}\int_t^{t+T} |u^{\mathsf{T}}(\tau)\phi(\tau)| d\tau \geq & \int_t^{t+T} |u^{\mathsf{T}}(\tau)\phi(t)| d\tau - \\ & \int_t^{t+T} |u^{\mathsf{T}}(\tau)[\phi(t) - \phi(\tau)]| d\tau.\end{aligned} \tag{5}$$

Using the bound in (3) the first integral on the righthand side of the above inequality can be bounded as

$$\int_t^{t+T} |u^\top(\tau)\phi(t)|d\tau \geq \|\phi(t)\|\frac{\alpha}{u_{\max}}. \tag{6}$$

The second integral on the righthand side of (5) can be bounded as

$$\begin{aligned}\int_t^{t+T} &|u^\top(\tau)[\phi(t)-\phi(\tau)]|d\tau \\ &\leq u_{\max}T \sup_{\tau\in[t,t+T]}\|\phi(t)-\phi(\tau)\| \\ &\leq u_{\max}T\int_t^{t+T}\|\dot\phi(\tau)\|d\tau \\ &\leq u_{\max}^2 T\int_t^{t+T}\|u^\top(\tau)\phi(\tau)\|d\tau.\end{aligned} \tag{7}$$

The second line in the above inequality follows by the fact that the arc-length between two points in space is always greater than or equal to a strait line between them. The third line in the above inequality follows by substition of the dynamics in (1). Substitution of the inequalities in (5)-(7) into (4) it follows that

$$\int_t^{t+T} |u^\top(\tau)\phi(\tau)|d\tau \geq \frac{\|\phi(t)\|\frac{\alpha}{u_{\max}}}{1+u_{\max}^2 T}.$$

Substitution of the above bound into (4) and squaring both sides it follows that

$$V(t+T)\leq\left(1-\frac{2\alpha^2/u_{\max}^2}{T(1+u_{\max}^2T)^2}\right)V(t).$$

Therefore the dynamics in (1) are UASL and by linearity this implies ESL as well. ■

While the first proof is more generic, the brute force method deployed in the second proof gives direct insight as to how the degree of persistency of excitation $\alpha$ and the upper bound $u_{\max}$ affect the rate of convergence

$$r_{\text{con}} \triangleq \left(1-\frac{2\alpha^2/u_{\max}^2}{T(1+u_{\max}^2T)^2}\right). \tag{8}$$

In the method by Anderson the rate of convergence is an existence one given by $(1-2\alpha_2)$. No closed form expression is given relating $\alpha_2$ to the original measures of PE, $\alpha$ and $\beta$.[1] It is clear however that for fixed $T$ an increase in $u_{\max}$ conservatively implies an increase in $\beta$. It is also clear from (8) that an increase in $u_{\max}$ decreases the convergence rate. It is comforting therefore to know that an increase in $\beta$ also implies a decrease in the convergence rate, which is now shown. Recall the Abel-Jacobi-Liouville identity, $\det\Phi_2(t,t_0)=\exp\left[-\int_{t_0}^t \text{trace}(u(\tau)u^\top(\tau))\,d\tau\right]$, and thus as $\beta$ increases, $\det\Phi_2(t,t_0)$ decreases. Now using this fact and the bound in (2) it follows that as $\beta$ increases $\alpha_2$ decreases. Thus the two proofs support the conclusions of the other and highlight one of the non-trivial characteristic of persistence of excitation and UASL [20, §6.5.3(a)].

[1] If one carefully follows the steps outlined in [1] it may be possible to come up with a closed form relation, but it appears to be non-trivial.

Returning to the main point of this note we consider a system were the degree of PE is a function of the initial conditions of the system. Consider the system

$$\dot\phi(t) = -u(t,\phi)u^\top(t,\phi)\phi(t) \tag{9}$$

with $\phi_0=\phi(t_0)$.

**Theorem 3.** *Let* $\Omega(r)=\{\phi:\|\phi\|\leq r\}$. *If* $u(t,\phi)\in$ *PE*$^*(\phi,\Omega(r))$ *for all* $r$ *and there exists* $u_{\max}(r)>0$ *such that* $\|u(t,\phi)\|\leq u_{\max}$ *for all* $\phi_0\in\Omega(r)$*, then* $\phi$ *in equation* (9) *is UASL and it* **does not follow** *that* (9) *is ESL.*

*Proof:* Given that $u(t,\phi)\in\text{PE}^*(\phi,\Omega(r))$, it follows that there exists $T(r)$ and $\alpha(r)$ such that $\int_t^{t+T}|u^\top(\tau,\phi)w|^2d\tau\geq\alpha$ for all $\phi_0\in\Omega(r)$. Choosing a Lyapunov candidate as $V(\phi,t)=\frac{1}{2}\phi^\top(t)\phi(t)$ and following the same steps as in the proof of Theorem 2 it follows that $V(t+T(r))\leq r_{\text{con}}V(t)$ for all $\phi_0\in\Omega(r)$ where

$$r_{\text{con}}(r)=\left(1-\frac{2\alpha^2(r)/u_{\max}^2(r)}{T(r)(1+u_{\max}^2(r)T(r))^2}\right).$$

Given that the convergence rate is upper bounded for all $\|\phi_0\|\leq r$ and $r$ can be arbitrarily large, the dynamics in (9) are UASL. Due to the fact that $r_{\text{con}}$ is not uniform in $\phi_0$ it can not be concluded that the dynamics are ESL. If for instance $r_{\text{con}}$ increases as $\|\phi_0\|$ increases then either $\kappa$ or $\nu$ in the definition of ESL would have to be increased or decreased respectively, thus breaking the definition of interest. ■

This is still a rather fictitious scenario similar to Example 1. In the next section we introduce the direct adaptive control problem, where this scenario is precisely what is encountered, the degree of persistence of excitation becomes initial condition dependent and thus the convergence rate will become initial condition dependent as well.

*Direct Model Reference Adaptive Control*

Let $u:[t_0,\infty)\to\mathbb{R}$ be the input and $x:[t_0,\infty)\to\mathbb{R}^n$ the state of a dynamical system

$$\dot x(t)=Ax(t)-B\theta^\top x(t)+Bu(t)$$

where $A\in\mathbb{R}^{n\times n}$ is known and Hurwtiz and $B\in\mathbb{R}^n$ is known as well, with the parameter $\theta\in\mathbb{R}^n$ unknown. The goal is to design the input so that $x$ follows a reference model state $x_m:[t_0,\infty)\to\mathbb{R}^n$ defined by the linear system of equations

$$\dot x_m(t)=Ax_m(t)+Br(t)$$

where $r:[t_0,\infty)\to\mathbb{R}$ is the reference command. Defining the model following error as $e=x-x_m$ the control input $u(t)=\hat\theta^\top(t)x(t)+r(t)$ achieves this goal when the adaptive parameter $\hat\theta:[t_0,\infty)\to\mathbb{R}^n$ is updated as follows

$$\dot{\hat\theta}(t)=-xe^\top PB$$

where $P=P^\top\in\mathbb{R}^{n\times n}$ is the positive definite solution to the Lyapunov equation $A^\top P+PA=-Q$ for any real $n\times n$ dimensional $Q=Q^\top>0$. So as to simplify the notation we let $C \triangleq PB$ and the adaptive system can be compactly represented as

$$\begin{bmatrix}\dot{e}(t)\\ \dot{\phi}(t)\end{bmatrix} = \begin{bmatrix} A & Bx^\top(t)\\ -x(t)C^\top & 0\end{bmatrix}\begin{bmatrix}e(t)\\ \phi(t)\end{bmatrix} \tag{10}$$

where the initial conditions of the model following error and parameter error are denoted as $e_0 = e(t_0)$ and $\phi_0 = \phi(t_0)$. For the dynamics of interest it follows that $V(e,\phi) = e^\top Pe + \phi^\top\phi$ is a Lyapunov candidate with time derivative along the state trajectories satisfying the inequality, $\dot{V} \le -e^\top Qe$. This implies that $e(t)$ and $\phi(t)$ are bounded for all time with

$$\|e\| \le \sqrt{V(e_0,\phi_0)/P_{\min}} \text{ and } \|\phi\| \le \sqrt{V(e_0,\phi_0)} \tag{11}$$

where $P_{\min}$ is the minimum eigenvalue of $P$. The reference command is bounded by design and thus $x_m$ is bounded and along with the bounds above implies that $x$ is bounded. The boundedness of $x$ and $\phi$ in turn implies that $\dot{e}$ is bounded for all time. Integration of $\dot{V}$ shows that $e \in \mathcal{L}_2$ with

$$\|e\|_{\mathcal{L}_2} \le \sqrt{V(e_0,\phi_0)/Q_{\min}} \tag{12}$$

where $Q_{\min}$ is the minimum eigenvalue of $Q$. From the fact that $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e} \in \mathcal{L}_\infty$ it follows that $e \to 0$ as $t \to \infty$ [20, Lemma 2.12]. Before discussing the asymptotic stability of the dynamics in (10) the following lemma is critical in relating persistence of excitation between the reference model state and the plant state. Let $z = [e^\top, \phi^\top]^\top$, then the dynamics in (10) can be compactly expressed as

$$\dot{z}(t) = \begin{bmatrix} A & Bx^\top(t,z;t_0)\\ -x(t,z;t_0)C^\top & 0\end{bmatrix} z(t) \tag{13}$$

where we have explicitly denoted $x$ as a function of the state variable $z$.

**Lemma 4.** *For the dynamics in* (13) *if $x_m(t)$ is PE, there exists $\alpha$ and $T$ such that $\int_t^{t+T} x_m(\tau)x_m^\top(\tau)d\tau \ge \alpha I$, and there exists a $\beta$ such that $\|x_m(t)\| \le \beta$, then $x(t,z)$ is PE$^*(z, Z(\zeta))$ with $Z(\zeta) = \{z : V(z) \le \zeta\}$ for all $\zeta > 0$ with the following bounds holding*

$$\int_t^{t+pT} x(\tau)x^\top(\tau)d\tau \ge \alpha' I \tag{14}$$

*with $p > p_{\min}$ where*

$$\sqrt{p_{\min}} \triangleq \frac{\left(\sqrt{\frac{\zeta}{P_{\min}}} + 2\beta\right)\sqrt{T\frac{\zeta}{Q_{\min}}}}{\alpha} \tag{15}$$

*and*

$$\alpha' \triangleq p\alpha - \left(\sqrt{\frac{\zeta}{P_{\min}}} + 2\beta\right)\sqrt{pT\frac{\zeta}{Q_{\min}}}. \tag{16}$$

Before going to the proof of this lemma a few comments are in order. First, note that the state variable $z$ contains both the model following error $e$ and the parameter error $\phi$. Therefore, what is being said is that $x$ is weakly persistently exciting for all initial conditions $e_0$ and $\phi_0$ in the compact regions defined by the level sets of the Lyapunov function $V(z) = e^T Pe + \phi^T\phi$. Furthermore, because these conditions hold for arbitrarily large level sets, i.e. $\zeta$ can be arbitrarily large, $x$ is weakly persistently exciting for any initial condition $z_0 \in \mathbb{R}^{2n}$. However, because the parameters in the persistence of excitation bound in (14), namely $p$, are not uniform in $z_0$ it can not be concluded that $x$ is PE.

*Proof:* This proof follows closely that of [4, Theorem 3.1]. For any fixed unitary vector $w$, consider the following equality, $(x_m^\top w)^2 - (x^\top w)^2 = -(x^\top w - x_m^\top w)(x^\top w + x_m^\top w)$. Using the definition of $e$, the bound in (11) for $e$ and the bound $\beta$ in the statement of the lemma, it follows that

$$(x_m^\top w)^2 - (x^\top w)^2 \le \|e\|(\sqrt{V(z_0)/P_{\min}} + 2\beta).$$

Moving $(x_m^\top w)^2$ to the righthand side, multiplying by $-1$ and integrating from $t$ to $t + pT$ where $p$ is defined just above (15)

$$\int_t^{t+pT}(x^\top(\tau)w)^2 d\tau \ge \int_t^{t+pT}(x_m^\top(\tau)w)^2 d\tau - \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2\beta\right)\int_t^{t+pT}\|e(\tau)\|d\tau.$$

Applying Cauchy-Schwartz to the integral on the right hand side and using the fact that $\int_t^{t+T}(x_m^\top(\tau)w)^2 d\tau \ge \alpha$ we have that

$$\int_t^{t+pT}(x^\top(\tau)w)^2 d\tau \ge p\alpha - \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2\beta\right)\sqrt{pT\int_t^{t+pT}\|e(\tau)\|^2 d\tau}.$$

Applying the bound in (12) for the $\mathcal{L}_2$ norm of $e$, it follows that

$$\int_t^{t+pT}(x^\top(\tau)w)^2 d\tau \ge p\alpha - \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2\beta\right)\sqrt{pT\frac{V(z_0)}{Q_{\min}}}.$$

For all $z_0 \in Z(\zeta)$ it follows that $V(z_0) \le \zeta$ and therefore

$$p\alpha - \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2\beta\right)\sqrt{pT\frac{V(z_0)}{Q_{\min}}} \ge \alpha'.$$

It follows directly that $\int_t^{t+pT}(x^\top(\tau)w)^2 d\tau \ge \alpha'$ for all $t \ge t_0$ and $z_0 \in Z(\zeta)$. ■

**Remark 1.** The main take away from this lemma is that for a given $\alpha$ and $T$ such that $\int_t^{t+T} x_m(\tau)x_m^\top(\tau)d\tau \ge \alpha I$ and for a fixed $\alpha'$ such that $\int_t^{t+pT} x(\tau)x^\top(\tau)d\tau \ge \alpha' I$, as the size of the level set $V(z) = \zeta$ is increased, $p$ must also increase. This can be seen directly through (15) where $p_{\min}$ increases with increasing $\zeta$. Thus, as $p$ increases, the time window over which the excitation is measured $pT$ increases as well.

**Theorem 5.** *If $x_m(t)$ is PE and uniformly bounded, then the dynamics in* (13) *are UASL and it* **does not follow** *that they are ESL.*

*Proof:* Given that $x_m \in$ PE it follows from Lemma 4 that $x(t,z) \in$ PE$^*(z, Z(\zeta))$ for any $\zeta$. For any fixed $\zeta$ applying [17, Theorem 5] implies that the dynamics of interest are UAS. Given that the above results hold for any $\zeta > 0$, the dynamics of interest are therefore UASL. Due to the fact that persistence of excitation bounds for $x$ do not hold globally uniformly in the initial condition $z_0$ one is not able to conclude ESL. ■

A simulation example illustrating the discussion in Remark 1 is now presented. Figure 1 below shows the state error $e$, the plant state $x$, the reference model state $x_m$, and the parameter error $\phi$ of a first order plant for three different initial conditions. The three different simulations going from solid line to dash-dot line to dotted line correlate to increasing $|\phi_0|$. The initial condition of the reference model is selected in conjunction with the reference input to remain constant for all time and thus $x_m(t) = 3$. As can be seen in the lower left plot of Figure 1 as $|\phi_0|$ increases, the amount of time that $x$ is close to 0 increases as well. This example illustrates the fact that while $x_m$ is persistently exciting $x$ is only weakly persistently exciting and shows the slow and non-exponential convergence of the errors as the initial conditions of the system increase. More examples of adaptive systems exhibiting this slow convergence can be found in [8, 9].

## V. Conclusion

Precise definitions of exponential stability along with initial condition dependent definitions of persistence of excitation have been given. With these definitions it has been shown that when the reference model is persistently exciting the direct adaptive control problem can at best be uniformly asymptotically stable in the large.

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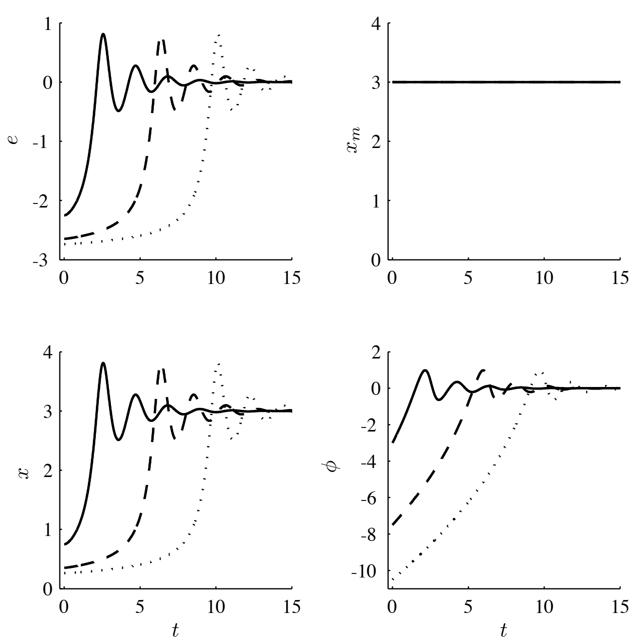


Fig. 1. Simulation of the the adaptive system in (10) with $A = -1, B = 1, P = 1, \theta = -2, r = 3, x_m(0) = 3$ and with the initial condition for $e$ and $\phi$ as illustrated in the figure.

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## Appendix

*Proof of Theorem 1:* Let $s(t; x_0, t_0)$ be a solution to the scalar differential equation of interest, then $s(t; x_0, t_0) = \mathbf{e}^{\int_{t_0}^{t} a(\tau)d\tau} x_0$. Let $\delta \triangleq t - kT - t_0$ where $k$ is the largest integer such that $\delta < T$. Using the integer $k$ and the parameter $T$ the time integral can be partitioned as

$$s(t; x_0, t_0) = \mathbf{e}^{\left(\int_{t_0}^{t_0+kT} + \int_{t_0+kT}^{t}\right) a(\tau)d\tau} x_0.$$

Multiplying both sides by $\mathbf{e}^{-\alpha_1(t-t_0)+\alpha_1(t-t_0)}$ we have that

$$s(t;x_0,t_0) = \mathbf{e}^{\left(\int_{t_0}^{t_0+kT} + \int_{t_0+kT}^{t}\right)a(\tau)d\tau} \cdot \mathbf{e}^{-\frac{\alpha_1}{T}(t-t_0)+\frac{\alpha_1}{T}(t-t_0)}x_0. \quad (17)$$

Noting that the bound $\int_t^{t+T} a(\tau)d\tau \leq -\alpha_1$ is uniform in time, it follows that $\int_{t_0}^{t_0+kT} a(\tau)d\tau \leq -\alpha_1 k$. Using the bound just derived and the definition of $\delta$ the following bound holds

$$\int_{t_0}^{t_0+kT} a(\tau)d\tau + \frac{\alpha_1}{T}(t-t_0) \leq \alpha_1\frac{\delta}{T} \leq \alpha_1 \quad (18)$$

where for the last inequality we have used the fact that $\delta$ is strictly less than $T$. Substitution of the bound in (18) into the equality in (17) results in the following

$$|s(t;x_0,t_0)| \leq \mathbf{e}^{\alpha_1}\mathbf{e}^{\int_{t_0+kT}^{t} a(\tau)d\tau}\mathbf{e}^{-\frac{\alpha_1}{T}(t-t_0)}|x_0|.$$

Finally, using the bound $a(t) \leq \alpha_2$ and the fact that $\delta < T$, it follows that $\mathbf{e}^{\int_{t_0+kT}^{t} a(\tau)d\tau} \leq \mathbf{e}^{\alpha_2 T}$ and thus

$$|s(t;x_0,t_0)| \leq \mathbf{e}^{\alpha_1+\alpha_2 T}\mathbf{e}^{-\frac{\alpha_1}{T}(t-t_0)}|x_0|.$$