Announcements

- Lab 2 is due this Wednesday, April 11, 2007.
- Problem set 7 is due this Friday, April 13, 2007.
- Quiz 2 will be held on Tuesday, April 24, 7:30–9:30 p.m. in Walker Memorial. Please let your TA, recitation instructor, and Xuemin know about any time conflicts right away.

Today’s Agenda

- Introduction to Sampling
- Sampling of CT Signals
  - CT impulse sample train
  - DT sample sequence
- Reconstruction of CT Signals: The Nyquist Sampling Theorem
  - Reconstruction from the CT impulse sample train
  - Reconstruction from the DT sample sequence
  - Viewing reconstruction in the time domain
- Summary and Subtleties of DT Sampling of CT Signals
- DT Processing of CT Signals
- CT Processing of DT Signals
  - Non-integer time shifting of DT signals
- Zero- and First-Order Holds
1 Introduction to Sampling

Sampling is our first introduction to combining CT and DT signals and systems. So far, we were able to distinguish between CT and DT signals because the variables for time were different: CT uses $t$ and DT uses $n$. Although both used $\omega$ for frequency, this was never a problem because we were always clear as to whether we were in CT or DT. However, we will be dealing with both CT and DT signals now, so it will be useful to use lower-case omega ($\omega$) for CT frequency and upper-case omega ($\Omega$) for DT frequency. Unfortunately, this convention is not standard. Sometimes$^1$, the notation is reversed!

2 Sampling of CT Signals

Consider the following block diagram, which consists of a continuous-to-discrete, or C/D converter and a corresponding discrete-to-continuous, or D/C converter, each with sampling period $T$ and corresponding sampling (angular) frequency $\omega_s = 2\pi/T$. We have a CT signal $x_c(t)$ being converted into the DT signal $x_d[n]$, which is then converted into the reconstructed CT signal $x_r(t)$:

The C/D converter sets the value of the DT signal to the value of the CT signal at integer multiples of $T$:

$$x_d[n] = x_c(t)|_{t=nT} = x_c(nT).$$

We will look later at how the D/C converter works.

2.1 CT impulse sample train

Let’s decompose the C/D converter into a multiplication with an impulse train $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$ and an impulse-to-sequence, or I/S, component. We call the intermediate CT signal $x_p(t) = x_c(t)p(t)$. Keep in mind that this decomposition is strictly a mathematical idealization. This helps us understand how the signals and their transforms evolve through the system, even though the actual physical implementation of such systems may be different:

$^1$such as in 6.341 Discrete-Time Signal Processing.
If \( x_c(t) \) is \textit{bandlimited}, meaning it only contains Fourier components within a finite frequency interval, then its CTFT \( X_c(j\omega) \) may look like:

\[
X_c(j\omega)
\]

The CTFT of an unit impulse train with spacing \( T \) is also an impulse train, but with spacing \( \omega_s \) and area \( 2\pi/T \) for each impulse. Since multiplication in time is convolution in frequency:

\[X_p(j\omega), \text{ the FT of } x_p(t), \text{ is repeated versions of } X_c(j\omega) \text{ scaled down by } T \text{ and spaced by } \omega_s.\]

This looks like:

\[
X_p(j\omega)
\]

2.2 \textbf{DT sample sequence}

We can then translate the CT impulse samples into DT samples using the I/S converter so that \( x_d[n] = x_c(nT) \). It can be shown (see textbook) that the DTFT of the DT sample sequence is related to the CTFT of the CT impulse sample train by:

\[
X_d(e^{j\Omega}) = X_p(j\Omega/T)
\]

So:

\[X_d(e^{j\Omega}), \text{ the FT of the DT sample sequence, is } X_p(j\omega), \text{ the FT of the CT impulse sample train, evaluated at } \omega = \Omega/T. \text{ This is simply a scaling of the frequency axis by } T.\]
3 Reconstruction of CT Signals: The Nyquist Sampling Theorem

We now have two types of samples of the original CT signal $x_c(t)$: a CT impulse sample train $x_p(t)$ and a DT sample sequence $x_d[n]$. In general, we cannot recover a signal from its samples, for an infinite number of signals share identical sample sequences. However, if we restrict ourselves to bandlimited signals, then if we sample at a sufficiently fast rate, the original signals can be recovered exactly. Let’s look at how we can produce a CT reconstruction $x_r(t)$ from each type of samples.

3.1 Reconstruction from the CT impulse sample train

Inspection of the frequency representation of the impulse sample train suggests using a low-pass filter with gain $T$ to retain the original FT pattern and restore its magnitude. This produces the following block diagram:

$$
\begin{array}{c}
x_p(t) \\
\xrightarrow{\text{Ideal LPF}} \\
x_c(t)
\end{array}
$$

However, this works only if there is no overlap, or aliasing, of the original FT spectrum, which occurs when $\omega_s$ is too low. This leads to the Nyquist Sampling Theorem:

**Nyquist Sampling Theorem:**

Let $x_c(t)$ be a bandlimited signal such that it has no frequency components for $|\omega| > \omega_M$. Then $x_c(t)$ can be uniquely reconstructed from its samples $x_c(nT)$ (or $x_d[n]$) if the sampling frequency satisfies $\omega_s > 2\omega_M$.

The sampling frequency must be strictly greater than twice the highest frequency present in the signal. The limit $2\omega_M$ is called the Nyquist rate.

3.2 Reconstruction from the DT sample sequence

Now that we solved the problem of obtaining $x_r(t)$ from the CT impulse sample train, we can just convert the DT sample sequence back into that CT impulse sample train and continue the reconstruction process as done before. To do this, we conceptually decompose the D/C converter into a sequence-to-impulse, or S/I, converter followed by the ideal reconstruction filter above, namely the low-pass filter. This entire block is also known as the ideal reconstruction system:
All the steps we went through before now occur in reverse, so, if we sample the bandlimited signal fast enough, the reconstruction \( x_r(t) \) is the same as the original signal \( x_c(t) \).

### 3.3 Viewing reconstruction in the time domain

We found it straightforward to analyze bandlimited reconstruction in the frequency domain. But what does it look like in the time domain? Since the reconstruction filter \( H_r(j\omega) \) is a low-pass filter (LPF), or a square pulse in frequency, its impulse response \( h_r(t) \) is a sinc in time.

We have “pseudo-convolution” in time between the DT sample sequence \( x_d[n] \) and the impulse response \( h_r(t) \) to obtain \( x_r(t) \). We’ll call it “pseudo-convolution” because we are “convolving” a DT signal with a CT one:

\[
x_r(t) = \sum_{n=-\infty}^{+\infty} x_d[n] h_r(t - nT)
\]

where

\[
h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}
\]

This leads to a relationship between the time representations of the DT sample sequence and the bandlimited reconstructed signal:

**The Time-Domain Interpolation Formula:**

The CT bandlimited interpolation \( x_r(t) \) is constructed from the DT sample sequence \( x_d[n] \) with sample period \( T \) using the formula:

\[
x_r(t) = \sum_{n=-\infty}^{+\infty} x_d[n] \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}
\]

This is one of the most interesting results in 6.003! Let’s examine it further to see what it really means. Suppose \( x_d[n] \) were a shifted impulse \( \delta[n - n_0] \). Then, the summation would only “sift out” the value of this expression at \( n = n_0 \). So:

A DT shifted impulse \( \delta[n - n_0] \) interpolates with sampling period \( T \) to the CT sinc function \( \frac{\sin(\pi(t - n_0 T)/T)}{\pi(t - n_0 T)/T} \), which has its peak at \( t = Tn_0 \) and is zero when \( t \) is any other integer multiple of \( T \).
Because any DT signal can be represented as the linear combination of shifted impulses, linearity implies that the interpolation formula can be interpreted as:

The Time-Domain Interpolation Formula in Plain English:

The CT bandlimited interpolation of a DT sample sequence with sampling period \( T \) is done by replacing each DT impulse with a CT sinc of the same height with zeros at integer DT times and superimposing the sincs.

So now, we understand exactly what it means to interpolate a DT signal to a CT one. But why does it makes sense that the CT bandlimited interpolation of a DT impulse is a sinc? Why isn’t the CT version of the DT impulse a CT impulse? Well, “bandlimited” means we want to use as low frequencies as possible to build the CT signal. The only way to get a CT signal that happens to hit zero at all the other integer multiples of \( T \) and still remain bandlimited is to use a sinc. Another way of thinking about this is to go back to the frequency domain. The FT of an impulse (in both DT and CT) is a constant, so it contains all frequencies in equal amounts. To translate this DTFT into the CTFT of a bandlimited signal, we need to truncate the high-frequency parts, leaving a centered low-frequency rectangular pulse, which we know is a sinc in time.

4 Summary and Subtleties of DT Sampling of CT Signals

We looked at the following system, which converts a CT signal \( x_c(t) \) into a DT signal \( x_d[n] \) (sampling), then back into a reconstructed CT signal \( x_r(t) \) (interpolation).

Sampling was a straightforward operation: \( x_d[n] = x_c(t) \). However, it was not obvious how to reconstruct a CT signal from a DT sample sequence. We found, according to the Nyquist sampling theorem, that if \( x_c(t) \) is bandlimited, or its Fourier transform is zero for frequencies above some \( \omega_M \), and if we sampled at least twice the highest frequency present, then we could reconstruct \( x_c(t) \) perfectly using bandlimited interpolation. This process was most easily visualized in the frequency domain. To aid in this process, we mathematically decomposed the C/D and D/C processes into two steps each: C/D is a CT impulse sampling followed by impulse-sequence (I/S) conversion and D/C is a S/I conversion followed by lowpass filtering.

We can summarize the frequency picture:

1. **CT bandlimited signal to CT impulse sample train**: CTFT is replicated at every integer multiple of \( \omega_s \) and is scaled down by \( T \).
2. **CT impulse train to DT sample sequence**: Frequency axis is scaled by \( T \): \( \Omega = \omega T \). Height is unchanged.
3. **DT sample sequence to CT impulse sample train**: Frequency axis is scaled by \( 1/T \): \( \omega = \Omega / T \). Height is unchanged.
4. **CT impulse train to CT bandlimited signal**: CTFT is lowpass filtered with cutoffs at \( \pm \omega_s / 2 \) and gain \( T \).

Since the breakdown of the C/D and D/C blocks were simply aids to help us develop the previously mentioned relationships, we can skip the CT impulse sample train completely and jump straight from a CT bandlimited signal to the DT sample sequence, and vice versa.
Another thing to watch out for:

**Linearity and Time-Variance of DT Processing of CT Signals:**

DT processing of CT signals is *always a linear process*, but is, in general, *time-varying* (not time-invariant). However, if the input CT signals are restricted to be bandlimited and the sampling rate is faster than Nyquist, then the overall CT system is LTI. These two are *sufficient but not necessary* conditions.

Thus, it is possible to sample slower than Nyquist yet still have an overall LTI system.

## 5 DT Processing of CT Signals

By inserting a DT system in our original C/D-D/C block diagram, we obtain a mechanism to perform *DT processing of CT signals*:

![Block Diagram of DT Processing of CT Signals](image)

Although this is in general a time-varying system, under certain conditions, then the entire system is a CT LTI system. The following describes this remarkable result:

**The Effective CT Frequency Response of DT Systems:**

If \( x_c(t) \) is bandlimited and the sampling rate is above the Nyquist rate, then the entire CT system is LTI, and the *effective* frequency response of the CT system \( H_c(j\omega) \) above is related to the frequency response of the DT system \( H_d(e^{j\Omega}) \) by:

\[
H_c(j\omega) = \begin{cases} 
  H_d(e^{j\omega T}), & |\omega| \leq \frac{\pi}{T} = \frac{\omega_s}{2} \\
  0, & |\omega| > \frac{\pi}{T}
\end{cases}
\]

Thus, we replace \( \Omega \) in the DT frequency response with \( \omega T \) and restrict the frequency range to the frequency spectrum of the bandlimited CT signal. It also turns out that:
Impulse-Invariance of CT and DT Systems:

The impulse response $h_d[n]$ of the DT system is a scaled, sampled version of the impulse response $h_c(t)$ of the CT system:

$$h_d[n] = T h_c(nT)$$

In other words, the DT system is an impulse invariant version of the CT system.

Many of the problems you’ll likely encounter on sampling will involve DT processing of CT signals. When working them out, try to keep the discreteness-periodicity duality in your mind. We’ll be switching back and forth between CT and DT, and also between time and frequency domains, so it’s helpful to use rules we developed before to double-check your progress.

As a reminder, here is the duality:

**The Discreteness-Periodicity Duality:**

Something that is discrete in one domain is periodic in the other domain, and something that is continuous in one domain is aperiodic in the other domain.
Problem 8.1

Let \( x_c(t) = \sin(75\pi t) / (\pi t) \). The signal \( x_c(t) \) is sampled every \( T \) seconds to produce the discrete time signal \( x_d[n] = x_c(nT) \). The signal \( x_d[n] \) is the input to a discrete LTI system with impulse response \( h_d[n] \), where \( h_d[n] = \sin(\pi n/4) / (\pi n) \). The output of the discrete time system \( y_p(t) \) is converted into an impulse train \( y_p(t) = \sum_{m=-\infty}^{\infty} y_d[m] \delta(t - mT) \). Finally the signal \( y_p(t) \) is the input to a low pass filter with impulse response \( h_L(t) = \sin(100\pi t) / (\pi t) \). The output of this low pass filter is \( y_c(t) \). For each of the cases below, find \( y_c(t) \).

(a) \( T = 1/50 \)

(b) \( T = 1/100 \)

(c) \( T = 1/200 \)

(d) \( T = 1/125 \)

(Workspace)
Problem 8.2  (From the 6.003 Spring 1999 Final Exam.)

Consider the following sampling system:

\[
x(t) \xrightarrow{C/D} x_d[n] \xrightarrow{h[n]} y_d[n] \xrightarrow{D/C} y(t)
\]

Figure 1: Sampling system.

The input \(x(t)\) has real-valued Fourier Transform \(X(j\omega)\) shown below.

\[
X(j\omega)
\]

Is it possible to adjust the sampling period \(T\) so that the Fourier transform \(Y(j\omega)\) of the output \(y(t)\) has the following shape?

\[
Y(j\omega)
\]

If yes, determine the sampling period \(T\) and the impulse response \(h[n]\) as a function of the parameters of \(X(j\omega)\). If no, explain why not.
Problem 8.3

Consider the standard system of DT processing of CT signals as shown in the previous problem, where the sampling period is $T$. Say the DT system satisfies the difference equation:

$$y_d[n] - \frac{1}{2}y_d[n - 1] + 3y_d[n - 2] = x_d[n].$$

(a) Find the frequency response $H(e^{j\Omega})$ of the DT system.

(b) Under what conditions does the entire system behave as a CT LTI system?

(c) Assuming these conditions are met, find the frequency response $H(j\omega)$ of the overall CT system.

(d) Assuming these conditions are met, find an equation relating $x(t)$ and $y(t)$. How does this equation relate to the difference equation above?

(Workspace)
Problem 8.4  (From the 6.341 Fall 2002 Background Exam)

Consider a system that performs DT processing of CT signals where the sampling frequency is 16 kHz. The DT filter is an ideal lowpass filter with a cutoff of $\pi/2$ rad/sample. What should the input $x_c(t)$ be bandlimited to so that the overall system is LTI?

(Workspace)
6 CT Processing of DT Signals

Likewise, we can also perform CT processing of DT signals by reversing the order of the C/D and D/C converters and inserting a CT system between them. This is not a practical thing to do in real-life, but it is an interesting exercise to think about to help us understand certain problems:

![Diagram of CT processing of DT signals]

However, a conceptual difficulty that doesn’t happen in the other direction arises: In DT processing of CT signals, we “took away” information by sampling. But in CT processing of DT signals, we are “adding in” extra information. What does it mean to create a CT signal from a DT one? After all, that DT signal is not a sequence of samples of some CT signal, so we are not reconstructing that CT signal. But the D/C converter doesn’t know that! So:

> The CT signal \( x_c(t) \) produced by a D/C converter from the DT signal \( x_d[n] \) is the bandlimited CT signal of which \( x_d[n] \) is the DT sample sequence. We can pretend that \( x_d[n] \) came from a bandlimited CT signal, and that the D/C converter outputs the reconstruction of this CT signal.

We saw that DT processing of CT signals are linear and time-varying, but the corresponding result is more straightforward for CT processing of DT signals:

**Linearity and Time-Invariance of CT Processing of DT Signals:**

CT processing of DT signals is always a linear time-invariant (LTI) process.

The impulse responses are also related: the DT impulse response is a scaled, sampled version of the CT impulse response, and the CT impulse response is a scaled, bandlimited interpolation of the DT impulse response, a relationship known as impulse invariance.² We’d like to emphasize that these statements reflect the equivalence of processing with and without C/D and D/C converters; there is no mathematical difference whatsoever.

Despite the usefulness of that equivalence, tricky problems tend to have different T’s for C/D and D/C and may also violate the Nyquist condition, so one would have to follow the transformation of the process step-by-step.

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²For the second part, we need to add the condition for CT processing DT signals that the embedded CT impulse response is appropriately bandlimited.
Similarly, we have:

**Effective DT Frequency Response of CT Systems:**

The *effective* frequency response of the entire DT system $H_d(e^{j\Omega})$ above is related to the frequency response of the CT system $H_c(j\omega)$ by:

$$H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right), \quad |\Omega| \leq \pi,$$

with $H_d(e^{j\Omega})$ periodic with period $2\pi$ (like all DTFTs).

This is a powerful result: this means that we can decompose *any* DT LTI system into a D/C converter followed by some CT LTI system followed by a C/D converter. This can help us understand using CT how some DT systems work. Non-integer time delay is a beautiful example of this.
Problem 8.5

Consider the standard system of CT processing of DT signals, where the sampling period is $T$. Say the CT system satisfies the differential equation:

$$\frac{d^2}{dt^2}y_c(t) + 2\frac{d}{dt}y_c(t) - y_c(t) = x(t)$$

- Find the frequency response $H(j\omega)$ of the CT system.
- Find the frequency response $H(e^{j\Omega})$ of the overall DT system. How can we interpret this?

(Workspace)
6.1 Non-integer time shifting of DT signals

The most elegant application of CT processing of DT signals is how to perform a non-integer time shift in DT signals. Back in chapter 1, we saw how to time delay a DT signal $x[n]$ by some integer amount $n_0$ to obtain $x[n - n_0]$. This process makes no sense when $n_0$ is not an integer. However, it would be nice if we had some way of interpreting a non-integer time delay of $\Delta$ to obtain “$x[n - \Delta]$.” The expression is in quotes because it is sloppy notation, but we know its intended meaning. As a first pass, we can try using zero- and first-order holds.

In this cases, these forms of non-integer delay are acceptable. However, let’s try to build the concept of non-integer delay by looking at a CT system that implements time delay in DT systems and is embedded between a D/C and a C/D converter. We begin with the frequency response we’d like to implement, which we can obtain by extrapolating the frequency for integer delay, which is:

$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta}, \quad |\Omega| \leq \pi.$$  

where $\Delta$ is an integer that indicates the shift in time. Thus, we expect the DT frequency response of a system that shifts a DT signal by non-integer $\Delta$ to be the same. Thus, the the frequency response $H_c(j\omega)$ of the embedded CT system is:

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta_c}, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

where

$$\Delta_c = T\Delta.$$  

This is a familiar system to us: it is the CT system that delays $x_c(t)$ by $\Delta_c$, when $x_c(t)$ is bandlimited to $\omega_c$. This leads to the following way of understanding non-integer delay:
Non-integer time delay in DT signals: We can interpret the time shifting of a DT signal $x[n]$ by any real (not necessarily integer) amount $\Delta$ to $y[n] = \text{“}x[n - \Delta]\text{”}$ as the DT sample sequence of the CT signal resulting from a time shift by $\Delta_c$ in the bandlimited CT interpolation of $x[n]$, where the CT time shift $\Delta_c$ is $\Delta_c = T\Delta$ and $T$ is the sampling rate. In other words, we can conceptually break down non-integer time delay of a DT signal into three steps:

1. Do a bandlimited interpolation into CT.
2. Shift the CT signal by the corresponding CT time delay.
3. Sample back into DT.

Mathematically, we have:

$$y[n] = x[n] * h[n],$$

where

$$h[n] = \frac{\sin \pi(n - \Delta)}{\pi(n - \Delta)}.$$

When $\Delta = n_0$ is an integer, then this reduces to $h[n] = \delta[n - n_0]$, which we recognize as the impulse response of a DT time shift system. Note that the impulse response of such a system is a DT sampled version of the CT sinc (impulse invariance).

At this point, one may ask, “Why use bandlimited interpolation? Why can’t I use zero- or first-order hold (discussed below) as my interpolation in the process of obtaining non-integer delay?” Well, only the bandlimited interpolation leads to a mathematically consistent analysis using Fourier transforms.

It can be shown that the Fourier transform of $h[n]$ is $H_a(e^{j\Omega})$, which is precisely what we need for this interpretation to be the “correct” one! In fact, this is a beautiful approach to finding the inverse Fourier transform of $e^{-j\Omega\Delta}$ for non-integer $\Delta$. In fact, this proves that our interpretation of non-integer time shifting as the DT sample sequence of the CT signal resulting from a time shift in the bandlimited CT interpolation of the original signal is the true method of performing this operation.

There is a slight conceptual difference in the approaches followed by the 6.003 textbook (pages 543 - 545) and the 6.341 textbook (pages 164 - 165) in developing this idea of non-integer delay. The former uses the DT processing of CT signals model. It asks what impulse response for the embedded DT system would produce a shift in the CT input by a non-integer multiple of $T$. The latter uses the CT processing of DT signals model. It asks what overall DT impulse response would give rise to the desired frequency response. Although it is not essential in 6.003 to know both, we encourage studying both to deepen your understanding of converting between CT and DT.
7 Zero- and First-Order Holds

We mentioned before that bandlimited interpolation is not the only conceivable method of producing a CT signal out of a DT sample sequence. The two simplest methods, the zero-order and first-order holds, are used all the time in some form or another. The zero-order hold “holds” the value of the sample for the duration of the sample period following the sample. Digital pictures and pixels are a form of zero-order hold. First-order hold is when we “connect the dots.”