Announcements

- Problem set 6 is due this Friday, April 6, 2007.
- Lab 2 is due next Wednesday, April 11, 2007!

Today’s Agenda

- Partial-Fraction Expansion
  - Top-heavy rationals
  - Repeated roots in the denominator
  - Canonical expansions
- First- and Second-Order Systems
- Bode Plots
  - First-order system Bode plots
  - Second-order system Bode plots
1 Partial-Fraction Expansion

Before we jump into the inverse Fourier transform, let’s discuss partial-fraction expansion, which is useful for rational Fourier transforms. Suppose \( F(x) \) is a rational function of \( x \) with the degree of the numerator strictly less than the degree of the denominator. That is:

\[
F(x) = \frac{(x - z_1)(x - z_2) \cdots (x - z_m)}{(x - p_1)(x - p_2) \cdots (x - p_n)},
\]

where \( m < n \) and \( z_i, p_j \) complex. If \( p_i \neq p_j \) for all \( i, j \), then \( F(x) \) may be written as:

\[
F(x) = \frac{A_1}{x - p_1} + \frac{A_2}{x - p_2} + \cdots + \frac{A_n}{x - p_n}.
\]

This is known as \textit{partial-fraction expansion}, or \textit{partial-fraction decomposition}. One way of finding the coefficients \( A_k \) is by multiplying through and matching terms. For instance, suppose:

\[
F(x) = \frac{2x + 1}{(x + 3)(x + 2)}
\]

Then,

\[
\frac{2x + 1}{(x + 3)(x + 2)} = \frac{A_1}{x + 3} + \frac{A_2}{x + 2}
\]

\[
2x + 1 = A_1(x + 2) + A_2(x + 3)
\]

\[
= (A_1 + A_2)x + (2A_1 + 3A_2)
\]

Matching the two coefficients gives:

\[
A_1 + A_2 = 2
\]

\[
2A_1 + 3A_2 = 1
\]

Solving these two equations produces the coefficients:

\[
A_1 = 5, \quad A_2 = -3
\]

However, solving systems of equations, even linear ones, can get rather tedious. A much faster method of finding the coefficients is as follows:

**Formula for the \( i \)th Expansion Coefficient:**

The coefficient \( A_i \) in Eq. 1 can be determined by:

\[
A_i = [(x - p_i)F(x)]|_{x=p_i}
\]
Let’s apply this analysis to our example and see why it works:

\[
A_1 = \left[ \frac{(x - p_1)F(x)}{(x + 3)(x + 2)} \right]_{x = p_1}^
\]

\[
= \left[ \frac{2x + 1}{(x + 3)(x + 2)} \right]_{x = -3}
\]

\[
= \frac{2x + 1}{x + 2} \bigg|_{x = -3}
\]

\[
\implies A_1 = 5
\]

We can proceed likewise for \( A_2 \). Note that the factors in the numerator and denominator cancel \textit{before} evaluation, which would have made that factor zero. Why does it work? Let’s multiply both sides of Eq. 7.1 by \((x + 3)\):

\[
\frac{2x + 1}{(x + 3)(x + 2)}(x + 3) = \frac{A_1}{x + 3} + \frac{A_2}{x + 2}(x + 3)
\]

\[
\frac{2x + 1}{x + 2} = A_1 + \frac{A_2}{x + 2}(x + 3)
\]

Now, we set \((x + 3)\) to zero, so that \textit{all terms except the one we want become zero}. Then, we have the value of \(A_1\). This is the algebra behind this trick. When we do it mechanically, it is known affectionately as “the cover up method.” This is because we tend to use our fingers to “cover up” the terms that disappear.

### 1.1 Top-heavy rationals

What do we do if the rational has a numerator whose order is equal to or higher than that of the denominator? We can use long division! For instance, suppose:

\[
F(x) = \frac{x^2 + 4x + 3}{x^2 + 4x - 5}
\]

Then,

\[
F(x) = \frac{(x^2 + 4x - 5) + 8}{x^2 + 4x - 5}
\]

\[
= 1 + \frac{8}{x^2 + 4x - 5}
\]

We can do a PFE on the right part of the sum:

\[
F(x) = 1 + \frac{8}{x^2 + 4x - 5}
\]

\[
= 1 - \frac{4/3}{x + 5} + \frac{4/3}{x - 1}
\]

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1.2 Repeated roots in the denominator

Sure, but what if we have repeated roots in the denominator? Specifically:

\[ F(x) = \frac{(x - z_1)(x - z_2) \cdots (x - z_m)}{(x - p_1)^{k_1}(x - p_2)^{k_2} \cdots (x - p_n)^{k_n}}, \]

where \( m < \sum_{i=1}^{n} k_i \) and \( z_i, p_j \) complex. Then, \( F(x) \) may be written as:

\[
F(x) = \frac{A_{i0}}{(x - p_1)^{k_i}} + \frac{A_{i1}}{(x - p_1)^{k_i-1}} + \cdots + \frac{A_{i}}{(x - p_1)} + \frac{A_{j0}}{(x - p_2)^{k_j}} + \frac{A_{j1}}{(x - p_2)^{k_j-1}} + \cdots + \frac{A_{j}}{(x - p_2)} + \cdots + \frac{A_{n0}}{(x - p_n)^{k_n}} + \frac{A_{n1}}{(x - p_n)^{k_n-1}} + \cdots + \frac{A_{n}}{(x - p_n)}
\]

The coefficients are:

**Formula for the Expansion Coefficient of the \( m \)th Factor and the \( (k_m - n) \)th Power:**

The coefficient \( A_{mn} \) in Eq. 7.1 can be determined by:

\[
A_{mn} = \frac{1}{n!} \left[ \frac{d^n}{dx^n} (x - p_m)^{k_m} F(x) \right]_{x=p_i}
\]

We’ll look at some examples of this later.

1.3 Canonical expansions

Particular partial fraction expansions come up so often that we should write them down once and for all and not have to resort to the cover up method each time. You may find the following expansions useful to put on an quiz sheet:

**Canonical Partial Fraction Expansions:**

\[
\begin{align*}
\frac{1}{(x + a)(x + b)} &= \frac{1}{b-a} - \frac{1}{a-b} \\
\frac{x}{(x + a)(x + b)} &= \frac{a}{a-b} + \frac{b}{b-a} \\
\frac{cx + d}{(x + a)(x + b)} &= \frac{a c - d}{a-b} + \frac{b c - d}{b-a}
\end{align*}
\]

Since this is a linear procedure, scaling the expressions on the left-hand side would also scale the expansions on the right-hand side by the same factor. Also, beware of the signs of \( a \) and \( b \)! You may want to jot down your own versions using \(-a\) and \(-b\).
Problem 7.1

Find the inverse DT Fourier transform \( x_i[n] \) for the following Fourier transform:

(a) \[
X_a(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}
\]

(b) \[
X_b(e^{j\omega}) = \frac{1}{(1 - \frac{1}{2}e^{-j\omega})^2(1 - \frac{3}{4}e^{-j\omega})}
\]

Find the inverse CT Fourier transform \( x_i(t) \) of the following Fourier transform:

(c) \[
X_c(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}
\]

(c) \[
X_d(j\omega) = \frac{(j\omega)^2 + 7}{(j\omega + 3)(j\omega + 4)}
\]

(Work space)
2 First- and Second-Order Systems

We can decompose a large LTI system into a combination of small LTI systems for which we know how to analyze. Our canonical systems are 1st and 2nd order systems. The following are the sets of descriptions for 1st and 2nd order systems:

1. 1st-order CT system

\[ \tau \dot{y}(t) + y(t) = x(t), \]

Freq. Resp. \[ H(j\omega) = \frac{1}{j\omega\tau + 1}, \]

Impulse Resp. \[ h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t). \]

\( \tau \) is known as the characteristic time constant for the system. An example of a first-order system is an RC circuit.

2. 1st-order DT system

\[ y[n] - ay[n-1] = x[n], \]

Freq. Resp. \[ H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, \]

Impulse Resp. \[ h[n] = a^n u[n]. \]

In a first-order DT system, \( a \) plays an analogous role to \( \tau \).

3. 2nd-order CT system

\[ \omega_n^2 x(t) = \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t), \]

Freq. Resp. \[ H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta \omega_n (j\omega) + \omega_n^2} = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}, \]

Impulse Resp. \[ h(t) = \begin{cases} \omega_n e^{-\zeta \omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) u(t) & c_1 \neq c_2, 0 < \zeta < 1 \\ M [e^{c_1 t} - e^{c_2 t}] u(t) & c_1 \neq c_2, \zeta = 1 \\ \omega_n^2 t e^{-\omega_n t} u(t) & c_1 = c_2 \end{cases} \]

where \( c_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \) and \( M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \). \( \omega_n \) is called undamped natural frequency and \( \zeta \) damping ratio. When \( \zeta > 1 \), both \( c_1 \) and \( c_2 \) are real, and we can decompose the system into two first-order systems, which we already know how to solve. The impulse response is then two decaying exponentials; we called this the overdamped case. When \( \zeta = 1 \), \( c_1 = c_2 \), and we have the critically damped case. Finally, when \( \zeta < 1 \), both \( c_1 \) and \( c_2 \) are complex, and we have the underdamped case, which has damped oscillations in its impulse response. Examples of second-order systems include an RLC circuit and a mass-spring-damper system.
3 Bode Plots

Let $H(j\omega)$ be the product of two transforms:

$$H(j\omega) = H_1(j\omega)H_2(j\omega) = |H_1(j\omega)||H_2(j\omega)|e^{\angle H_1(j\omega) + \angle H_2(j\omega)}$$

We can represent the magnitude and phase of $H(j\omega)$ separately:

$$|H(j\omega)| = |H_1(j\omega)||H_2(j\omega)|$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

If we take the logarithm of both sides of the magnitude equation and multiply by 20, we get:

$$20 \log |H(j\omega)| = 20 \log |H_1(j\omega)| + 20 \log |H_2(j\omega)|$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

These equations represent the log magnitude and phase of a frequency response as the sum of the corresponding components of the factors of the frequency response. Adding plots is easy, so this is a convenient way for use to build pictures of the frequency response if we can break it down into simpler components. The units of $20 \log$ are decibels (dB). In CT, we also use a log scale for frequency. We call plots of $20 \log |H(j\omega)|$ and $\angle H(j\omega)$ vs. log $\omega$ Bode plots. We will almost always deal with real systems, so $H(j\omega)$ is conjugate-symmetric. Therefore, we will only plot $H(j\omega)$ for positive $\omega$.

3.1 First-order system Bode plots

Let’s find the Bode plot for a first-order system:

$$H(j\omega) = \frac{1}{j\omega\tau + 1}.$$  

As $\omega$ runs from zero to infinity, the magnitude of the vector increases monotonically to infinity and the angle runs from zero to $\pi$. Analytically, we have:

$$20 \log |H(j\omega)| = -10 \log((\omega\tau)^2 + 1)$$

$$\angle H(j\omega) = -\tan^{-1}(\omega\tau).$$

When $\omega\tau << 1$, the log magnitude is near zero, and when $\omega\tau >> 1$, the log magnitude is

$$20 \log |H(j\omega)| = -10 \log((\omega\tau)^2 + 1) \approx -10 \log((\omega\tau)^2) = -20(\log \omega) - 20(\log \tau),$$

which is linear in log $\omega$ with slope -20. This line is called the 20-dB-per-decade asymptote. Thus, we can represent the log magnitude of the frequency response by two asymptotes: two straight lines for each regime that meet at the break frequency $1/\tau$. The actual plot deviates from the asymptotic approximation the most at the break frequency itself, when it is:

$$-20 \log \sqrt{2} = -10 \log 2 \approx -3 \text{dB}.$$
Likewise, we approximate the angle as being 0 for $\omega << 0.1/\tau$, $-\pi/2$ for $\omega >> 10/\tau$, and a line connecting the two extremes in between. At the break frequency, the asymptote and the actual value coincide: $\angle H(j/\tau) = -\pi/4$.

If we can factor a frequency response into first-order polynomials with real roots on both top and bottom, then we can apply the analysis above to each and add them up to form the Bode plot of the frequency response. However, it is important to rewrite the frequency response from pole-zero form:

$$H(j\omega) = M \frac{(j\omega - z_1)(j\omega - z_2) \cdots (j\omega - z_Q)}{(j\omega - p_1)(j\omega - p_2) \cdots (j\omega - p_P)},$$

to time-constant form:

$$H(j\omega) = K \frac{(j\omega)(\tau z_1 j\omega + 1)(\tau z_2 j\omega + 1) \cdots (\tau z_R j\omega + 1)}{(\tau p_1 j\omega + 1)(\tau p_2 j\omega + 1) \cdots (\tau p_P j\omega + 1)}.$$

The advantage is we can read off right away what the DC gain (or asymptote) is. The Bode plots can easily be constructed from each factor:

$$H(j\omega) = K \frac{(j\omega)(1 + \tau z_1 j\omega) \cdots (1 + \tau z_R j\omega)}{(1 + \tau p_1 j\omega) \cdots (1 + \tau p_P j\omega)}$$

$\Rightarrow \log |H(j\omega)| = \log |K| + L \log |\omega| + \sum_{i=1}^{R} \log |(1 + \tau z_i j\omega)| - \sum_{i=1}^{P} \log |(1 + \tau p_i j\omega)|$

$\Rightarrow \angle H(j\omega) = \angle K + L \angle(j\omega) + \sum_{i=1}^{R} \angle(1 + \tau z_i j\omega) - \sum_{i=1}^{P} \angle(1 + \tau p_i j\omega)$

From this form, we use the following procedure:
Building the CT Bode Plot from First-Order Polynomials:

Consider a rational transfer function in time-constant form, where each of the time constant is positive and real:

\[ H(j\omega) = K \frac{(j\omega)^L (\tau_{z1}j\omega + 1)(\tau_{z2}j\omega + 1) \cdots (\tau_{zR}j\omega + 1)}{(\tau_{p1}j\omega + 1)(\tau_{p2}j\omega + 1) \cdots (\tau_{pP}j\omega + 1)}. \]

The Bode plot of the frequency response \( H(j\omega) \) can be produced with the following rules:

- The constant gain \( K \) shifts the log magnitude up by \( 20 \log K \) dB.
- Factors in the numerator of the transfer function
  - A factor of \( j\omega \) contributes to the log magnitude a line that is 0 dB at \( \omega = 1 \) rad/s with slope 20 dB/dec and to the phase a constant of \( \pi \) (90 degrees).
  - A factor of \( (\tau_{z1}j\omega + 1) \) contributes to the log magnitude a bent line that is 0 dB for \( \omega < 1/\tau_{z1} \) (\( \omega \) is less than the break frequency) and has slope 20 dB/dec for \( \omega > 1/\tau_{z1} \) and to the phase a bent line that is 0 for \( \omega < (1/\tau_{z1})/10 \) (one decade below the break frequency), \( \pi/2 \) (90 degrees) for \( \omega > 10/\tau_{z1} \) (one decade above the break frequency), and is linear with slope \( \pi/4 \) rad/decade (45 degrees/decade) in between.
- Factors in the denominator of the transfer function
  - A factor of \( j\omega \) contributes to the log magnitude a line that is 0 dB at \( \omega = 1 \) rad/s with slope -20 dB/dec and to the phase a constant of \(-\pi\) (−90 degrees).
  - A factor of \( (\tau_{z1}j\omega + 1) \) contributes to the log magnitude a bent line that is 0 dB for \( \omega < 1/\tau_{z1} \) (\( \omega \) is less than the break frequency) and has slope -20 dB/dec for \( \omega > 1/\tau_{z1} \) and to the phase a bent line that is 0 for \( \omega < (1/\tau_{z1})/10 \) (one decade below the break frequency), \(-\pi/2\) (−90 degrees) for \( \omega > 10/\tau_{z1} \) (one decade above the break frequency), and is linear with slope \(-\pi/4\) rad/decade (−45 degrees/decade) in between.

Notice that this summary in words did not include the possibility of negative time constants. They work exactly the same way, except that the phase is flipped in sign. Generally, we will not see such factors in the denominator, for these would make a causal system unstable.
Problem 7.2

Draw the Bode plot for the following frequency response:

\[ H(j\omega) = \frac{10^5(j\omega + 10)}{j\omega(j\omega + 100)(j\omega + 10^4)} \]

(Work space)
3.2 Second-order system Bode plots

The standard form for the transfer function of a second-order system is

\[ H(s) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{\omega_n^2}{(j\omega - p_+)(j\omega - p_-)}. \]

where \( \omega_n \) is the undamped natural frequency and \( \zeta \) is the damping ratio. These terms come from typical second-order systems, such as RLC circuits and mass-spring-dashpot mechanical systems. The roots of the denominator (which we will refer to as poles after studying the Laplace transform) are

\[ p_{\pm} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}. \]

When \( \zeta > 1 \), the roots are real, and we decompose the system into two first-order systems, which we already know how to solve. The impulse response is then two decaying exponentials; we called this the overdamped case. When \( \zeta = 1 \), the roots are equal, and we have the critically damped case. Finally, when \( \zeta < 1 \), both roots are complex, and we have the underdamped case, which has damped oscillations in its impulse response. The textbook shows some Bode plots for various values of the damping ratio, which greatly affects the shape of the plots. For \( \zeta < \sqrt{\frac{2}{3}} \), there is a maximum peak in the log magnitude of the frequency response at

\[ \omega_{\text{max}} = \omega_n \sqrt{1 - 2\zeta^2}, \]

where the value is

\[ |H(j\omega_{\text{max}})| = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}. \]

The “quality factor” \( Q \) is commonly used to describe the sharpness of the peak at \( \omega = \omega_{\text{max}} \) and is given by

\[ Q = \frac{1}{2\zeta} \]

For a high-\( Q \) second-order system (such as a RLC circuit or a spring-mass-dashpot system) that begins with some energy and is allowed to evolve without further input, a useful energy interpretation of \( Q \) is \( 2\pi \) times the inverse of the proportion of energy lost per oscillation. Note that if \( \zeta << 1 \) (the undamped case, very high-\( Q \)),

\[ \omega_{\text{max}} \approx \omega_n, \quad |H(j\omega_{\text{max}})| \approx \frac{1}{2\zeta}. \]

If \( \omega \) moves to \( \omega_n \pm \zeta \omega_n \), magnitude is decreased by \( 1/\sqrt{2} \). Power is magnitude squared, so power is cut in half (3 dB lower). Hence, the margin \( 2\zeta \omega_n \) is called the half-power bandwidth, or the full-width half maximum when power is plotted. We also see that the angle in this range falls from \(-\pi/4\) to \(-3\pi/4\). For very low frequencies, the magnitude is 1 (0 dB) and the angle is zero. For very high frequencies, the magnitude becomes a -40 dB/dec line and the angle becomes \(-\pi\). This is best illustrated by an example problem.
Problem 7.3

Sketch the Bode magnitude and phase plots for the following frequency response:

\[ H(j\omega) = \frac{101}{(j\omega)^2 + 2j\omega + 101}. \]

(Work space)
Problem 7.4

We are given the following frequency responses:

\[ H_1(j\omega) = \frac{j\omega + 100}{10(j\omega + 10)} \]
\[ H_2(j\omega) = \frac{j\omega - 100}{10(j\omega + 10)} \]
\[ H_3(j\omega) = \frac{10(j\omega + 10)}{j\omega + 100} \]
\[ H_4(j\omega) = \frac{100}{(j\omega)^2 + 4j\omega + 100} \]
\[ H_5(j\omega) = \frac{j\omega + 100}{(j\omega)^2 + 4j\omega + 100} \]
\[ H_6(j\omega) = \frac{j\omega + 100}{10(j\omega + 10)} \]

(a) Match each of the frequency response functions above to its corresponding Bode magnitude plot.

(b) Draw the Bode phase plots for \( H_1(j\omega) \) and \( H_2(j\omega) \).

(c) Does there exist \( H_7(j\omega) \) such that \(|H_7(j\omega)| = |H_1(j\omega)|\) and \(|H_7(j\omega)| = |H_2(j\omega)|\), but \( \angle H_7(j\omega) \) is not identical to \( \angle H_1(j\omega) \) nor \( \angle H_2(j\omega) \)? If there exists, find an expression for \( H_7(j\omega) \). If not, explain why.

(d) Draw the Bode plots for the frequency responses for which there are not corresponding magnitude plots in (a).
(Work space)