Announcements

• Problem Set 5 due this Friday
• Quiz 1 will be returned in recitation on Wednesday.
• Quiz 1 solutions will be online this week.

Today’s agenda

• Discrete-Time Fourier Transform
• Similarities and differences to CTFT
• DTFT Properties
DTFT: Discrete-Time Fourier Transform  Like CT signals, we can write DT signals as the superposition of exponentials of a continuum of frequencies. This representation of DT signals is known as the Discrete-Time Fourier Transform. The relationships between a sequence, \( x[n] \) and its DTFT, \( X(e^{j\omega}) \) are given by

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \quad \text{Synthesis Eqn}
\]

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \quad \text{Analysis Eqn}
\]

The \( 2\pi \) in the integral indicates that we can choose any two limits, \( \omega_1, \omega_2 \) such that \( \omega_2 = \omega_1 + 2\pi \). We will see why any such pair of limits will be equivalent. For example:

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \\
= \frac{1}{2\pi} \int_{0}^{2\pi} X(e^{j\omega})e^{j\omega n} d\omega \\
= \frac{1}{2\pi} \int_{-0.123}^{-0.123+2\pi} X(e^{j\omega})e^{j\omega n} d\omega
\]

Typically, we will choose the range \( \omega \in [0, 2\pi] \) or \( \omega \in [-\pi, \pi] \). This is a major difference from the CT Fourier Transform, which required an integral from \( \omega = -\infty \) to \( \omega = +\infty \). The reason for this difference can be understood by examining the complex exponential \( e^{j\omega n} \).
**DT complex exponentials** The major differences between CTFT and DTFT arise due to the inherent differences between the CT $e^{j\omega t}$ and the DT $e^{j\omega n}$. We discussed these differences at the beginning of the course, but it is worthwhile to revisit and understand how these differences affect Fourier Transforms.

- In CT, the complex exponential is always periodic in time:

  $$e^{j\omega(t+T)} = e^{j\omega t} \quad \text{where} \ T = \frac{2\pi}{\omega} \text{ is the fundamental period}$$

  However, $e^{j\omega t}$ is not periodic in frequency. As we increase $\omega$, the oscillation of the sinusoids (remember real and imaginary parts are $\cos \omega t$ and $\sin \omega t$) gets faster and faster continuously with no end.

- In DT, the complex exponential $e^{j\omega n}$ again has sinusoidal real and imaginary parts, but not all $\omega$ will result in periodic sequences. As we discussed previously, it will be periodic only if the frequency is a rational multiple of $2\pi$, i.e. $\omega = \frac{2\pi M}{N}$ for any integers $N$ and $M$. For example, $\sin(\pi n)$ is periodic but $\sin(n)$ is not.

  More importantly, the DT complex exponential is **periodic in frequency** with a period of $2\pi$. This is easily seen by:

  $$e^{j(\omega+2\pi)n} = e^{j\omega n} e^{j2\pi n} = e^{j\omega n} \cdot 1^n = e^{j\omega n}$$

  So, if we continually increase $\omega$, the oscillations cannot get faster forever, instead things wrap around and we are back to where we started each time we add a multiple of $2\pi$.

  Thus, in DT, there is a highest frequency.

**Effect on DTFT** With this understanding of the DT complex exponential, it is easy to show that the DTFT is a periodic function with fundamental period $2\pi$.

  $$X(e^{j(\omega+2\pi)n}) = \sum_{n=-\infty}^{+\infty} x[n] e^{j(\omega+2\pi)n} = \sum_{n=-\infty}^{+\infty} x[n] e^{j\omega n} = X(e^{j\omega})$$

**DTFT Properties** Like CTFT, O&W provides tables of DTFT properties and pairs on pages 391 and 392. While most properties are equivalent to their CT counterparts, there are some subtle but important differences due to the periodicity of $X(e^{j\omega})$.

  The general approach to determining Fourier Transforms is same for both CT and DT:

  1. Write a given signal in terms of simple signals for which we know the Fourier Transform (i.e. you can look them up in a table - but be sure you can derive them if asked)
2. Use properties to relate the Fourier Transform of the given signal to those of the simple signals. (Again, you can often memorize or look up these properties, but understand where they come from.)
Problem 1  Compute the DTFT of the following signals using the analysis equation:

(a) \( x[n] = \left(\frac{1}{2}\right)^nu[n - 2] \)

(b) \( x[n] = 2^n \sin\left(\frac{\pi}{4}n\right)u[-n] \)
**DT LTI Systems**  Most of the concepts from CT systems and CTFT apply to DT systems and DTFT.

\[
x[n] \xrightarrow{\text{DT LTI}} y[n] = x[n] * h[n] \\
X(e^{j\omega}) \xrightarrow{\text{DT LTI}} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})
\]

When dealing with DT LTI systems, one of the most important properties will be **convolution property**. This is written below with its counterpart, the **multiplication property**:

- **DT convolution property**
  \[
x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})
  \]

- **DT multiplication property**
  \[
x[n]y[n] \leftrightarrow \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta
  \]

Notice that the convolution in the 2nd property is only over a region of \(2\pi\) since both \(X(e^{j\omega})\) and \(Y(e^{j\omega})\) are both periodic in \(\omega\) with period \(2\pi\). This is known as **periodic convolution**.
Problem 2: Periodic Convolution  Consider the following $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$.

1. What happens if you try to perform a standard convolution of these two signals?

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$

2. Now let’s perform a periodic convolution.

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2(e^{j\theta}) X_1(e^{j(\omega-\theta)}) d\theta$$

3. Is the resulting function guaranteed to be periodic with period $2\pi$?

4. How can we alternately view $Y(e^{j\omega})$ in terms of standard convolutions?
Problem 3. Consider a signal $x[n]$ with Fourier Transform $X(e^{j\omega})$ shown below:

Let $w[n] = p[n]x[n]$ where

$$p[n] = \cos \pi n - \cos \left( \frac{\pi n}{2} \right).$$

Sketch $W(e^{j\omega})$, the Fourier Transform of $w[n]$. 
Problem 4  Consider a DT LTI system with impulse response:

\[ h[n] = \frac{\sin(\frac{\pi n}{4})}{\pi n} \cdot (-1)^n \]

Suppose the input is:

\[ x[n] = \cos\left(\frac{\pi n}{8}\right) + \sin\left(\frac{\pi n}{2}\right) + (-1)^n. \]

Determine the output \( y[n] \).
Problem 5  Consider a DT LTI system with impulse response \( h[n] = \frac{1}{2}(u[n] - u[n - 2]) \)

1. Determine expressions for \( |H(e^{j\omega})| \) and \( \angle H(e^{j\omega}) \) and plot them. What type of filter is this?

2. Consider the input \( x_1[n] = (-1)^n \). Determine \( X_1(e^{j\omega}) \). Show that the output is zero using both convolution in the time domain and multiplication in the frequency domain.

3. Let \( x_2[n] = 1 \) for all \( n \). Find the output using time and frequency domain.

4. Find a simple transformation to turn \( h[n] \) to a high-pass filter and plot it’s frequency response. Determine outputs from \( x_1[n] \) and \( x_2[n] \).