Tutorial 12

Solutions

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Find $z$-transforms and ROCs

(a) $x_a[n] = 3\delta[n+4] - 5\delta[n+3] + 2\delta[n+2]$

if $x_a[n] = 3\delta[n+4]$:

\[
\text{Z-Transform: } X(z) = \sum_{n=-\infty}^{\infty} x_a[n] z^{-n} = 3 \sum_{n=-\infty}^{\infty} \delta[n+4] z^{-n} = 3 z^{-4} = 3z^{-4}
\]

converges for all $z$ except $z = \infty$

$\delta[n-m] \rightarrow z^{-m}$ ROC: All $z$ except $0$ (m>0)

or $\infty$ (if m<0)

So

\[
X_a(z) = 3z^{-4} - z^{-3} + 2z^{-2}
\]

ROC: All $z$ except $\infty$

(b) $x_b[n] = x_a[n-3] = 3\delta[n+1] - 5\delta[n] + 2\delta[n-1]$

Can use time-shifting property:

$X[n-n_0] \rightarrow z^{-n_0} X(z)$

\[
X_b(z) = z^{-3} X_a(z) = z^{-3} \left[ 3z^{-4} - z^{-3} + 2z^{-2} \right] = 3z^{-7} - z^{-6} + 2z^{-5}
\]

$X_b(z) = 3z^{-1} - 2z^{-4}$ ROC: All $z$ except $\infty$ and 0

$X_b(z)$ does not converge for $z = \infty$ because of the $3z$ term.

$X_b(z)$ does not converge for $z = 0$ because of the $2z^{-4}$ term.
12.1 (c) \( X_c[n] = 2^n u[n-1] + 4^n u[-n] \)

Method 1: \( z \)-transform

\[
X_c(z) = \sum_{n=0}^{\infty} (2)^n z^{-n} + \sum_{n=\infty}^{0} (4)^n z^{-n}
\]

\[
= \sum_{n=1}^{\infty} (2z^{-1})^n + \sum_{n=-\infty}^{0} (4z^{-1})^n
\]

\[
= \frac{(2z^{-1}) - (2z^{-1})^{\infty+1}}{1-2z^{-1}} + \frac{(4z^{-1}) - (4z^{-1})^{\infty}}{1-4z^{-1}},
\]

converges for 
\[ |2z^{-1}| < 1 \]

\[
= \frac{2z^{-1}}{1-2z^{-1}} + \frac{-4z^{-1}}{1-4z^{-1}} = \frac{2z^{-1} - 8z^{-2} - 4z^{-1} + 8z^{-2}}{(1-2z^{-1})(1-4z^{-1})}
\]

\[
= \frac{-2z^{-1}}{(1-2z^{-1})(1-4z^{-1})} = -\frac{2z}{(z-2)(z-4)}
\]

\[
\text{So } 2 < |z| < 4
\]

Method 2: Properties + Table

\( 2^n u[n-1] = 2 \left[ (2)^{-n} u[\neg n-1] \right] \)

using time shift property \( u[n-n] \leftrightarrow z^{-n}X(z) \)

\[
\frac{2z^{-1}}{1-2z^{-1}} \text{ ROC: } |z| > 2
\]

\( 4^n u[-n] = \left(\frac{1}{4}\right)^{-n} u[-n] \) time reversal property \( \text{ROC} \)

\[
\frac{1}{1-\left(\frac{1}{4}\right)z^{-1}} \text{ ROC: } |z| > \frac{1}{4}
\]

\( (\frac{1}{4})^n u[n] \rightarrow \frac{1}{1-\left(\frac{1}{4}\right)z^{-1}} \text{ ROC: } |z| > \frac{1}{4} \)

\[
X_c(z) = \frac{2z^{-1}}{1-2z^{-1}} \text{ ROC: } 2 < |z| < 4
\]

\[
= \frac{2z}{z-2} + \frac{-4}{z-4} = \frac{2z^2 - 8 - 4z + 8}{(z-2)(z-4)}
\]

\[
X_c(z) = \frac{-2z}{(z-2)(z-4)} \quad 2 < |z| < 4
\]
Method 1: \( Z \)-transform

\[
X_d(z) = \sum_{n=-\infty}^{\infty} (\frac{1}{2^n}) z^{-n} - (\frac{1}{2-z})^5 = \frac{\left(\frac{1}{2z^{-1}}\right)^{\infty} - \left(\frac{1}{2z^{-1}}\right)^5}{1 - \frac{1}{2z^{-1}}}
\]

Converges for

\[
|\frac{1}{2}z^{-1}| > 1 \quad \text{or} \quad |z| < \frac{1}{2}
\]

\[
= \frac{0 - \frac{1}{32}z^{-5}}{1 - \frac{1}{2}z^{-1}} \quad |z| < \frac{1}{2}
\]

\[
X_d(z) = \frac{-z^{-5}}{32(1-\frac{1}{2}z^{-1})} \quad |z| < \frac{1}{2}
\]

Method 2: Properties & Tables

\[
X_d[n] = (2)^{-n}u[-n+4]
\]

Time reversal property

\[
X[-n] \leftrightarrow X(z^{-1}) \quad \text{Inverted ROC}
\]

Time shift property

\[
X[n-m] \leftrightarrow z^{-m}X(z)
\]

So, with time shift:

\[
\frac{1}{16} (2)^{-n+4}u[-n+4] \xrightarrow{z^{-1}} \frac{-z^{-4}}{16(1-2z^{-1})} \quad \text{ROC: } |z| > 2
\]

with time reversal:

\[
\frac{1}{16} (2)^n u[-n+4] \xrightarrow{z^{-1}} \frac{z^{-4}}{16(1-2z^{-1})} \quad \text{ROC: } |z| < \frac{1}{2}
\]

\[
X_d(z) = \frac{z^{-5}}{16(z^{-1}-2)} = \frac{-z^{-5}}{16(2-z^{-1})} = \frac{-z^{-5}}{32(1-\frac{1}{2}z^{-1})} \quad |z| < \frac{1}{2}
\]
12.2: Semi-periodic $x[n]$:

- $x[n] = 0$ for $n < 0$
- $x[0] = 2$
- $x[1] = 3$
- $x[2] = -1$
- $x[n] = x[n-3]$ for $n > 2$

$x(z) = ? \quad$ ROC = ?

This can be thought of as the convolution between two signals

$$x_a[n] \quad x_b[n]$$

$$x[n] = \begin{cases} 2 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ \vdots \end{cases} \ast \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ \vdots \end{cases}$$

$x(z) = x_a(z) \cdot x_b(z)$

$x_a(z) = 2 + 3z^{-1} - z^{-2} \quad$ ROC: all $z$ except 0

Two ways to get $x_b(z)$:

1. $z$-transform:

$$x_b[n] = \sum_{k=0}^{\infty} x_b[n-3k]$$

$$x_b(z) = \sum_{n=0}^{\infty} x_b[n] z^{-n} = \sum_{n=0}^{\infty} z^{-3n} = \sum_{n=0}^{\infty} \left(z^{-3}\right)^n = \frac{1}{1 - z^{-3}}$$

2. Time expansion property:

We can recognize that $x_b[n]$ is a time expanded unit impulse function.

$$x_b[n] = U[n-\frac{3}{2}]$$

So $x_b[n] = U(n) \iff n \gg 0$ using $X_u[n] \leftrightarrow X(z)$

if $x_0[n] = U[n]$

$$X_u(z) = \frac{1}{1-z^{-1}}$$

$$x_0[n] \Rightarrow X_u(z) = \frac{1}{1-(\frac{3}{2})^{-1}} = \frac{1}{1-2^{-3}}$$

Thus

$$X(z) = x_a(z) \cdot x_b(z) = \frac{2 + 3z^{-1} - z^{-2}}{1 - 2^{-3}} \quad \text{ROC: } |z| > 1$$
\[ X(z) = \frac{4 + \frac{1}{2} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{3} z^{-1})} = \frac{4 + \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1} + \frac{1}{3} z^{-2}} = \frac{4 + \frac{1}{2} z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{3} z^{-1}\right)} \]

Determine \( X[n] = ? \)

ROC:
(i) \( |z| < \frac{1}{3} \)

Use PFE:
\[ X(z) = \frac{4 + \frac{1}{2} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{3} z^{-1})} = \frac{A}{1 - \frac{1}{2} z^{-1}} + \frac{B}{1 + \frac{1}{3} z^{-1}} \]

Method 1: Cover up (headdside)
\[ A = (1 - \frac{1}{2} z^{-1})X(z) \bigg|_{z^{-1} = 2} = \frac{4 + \frac{1}{2} z^{-1}}{2} - \frac{1}{3} \] \( \Rightarrow \) \( \frac{5}{2} \beta = \frac{5}{2} \beta = 1 \) \( \Rightarrow \) \( A = 3 \)

\[ \frac{5}{2} B = \frac{5}{2} \beta \Rightarrow B = \frac{5}{3} \]

So:
\[ X(z) = \frac{3}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} \]

\[ X[n] = -3(\frac{1}{2})^{n} u[-n-1] - (\frac{1}{3})^{n} u[-n-1] \] for \( |z| < \frac{1}{3} \)

\[ X[n] = u[-n-1] \] for \( |z| > \frac{1}{3} \)

(ii) \( \frac{1}{3} < |z| < \frac{1}{2} \)

So for first term \( |z| < \frac{1}{3} \) & second term \( |z| > \frac{1}{3} \)

\[ X[n] = -3(\frac{1}{2})^{n} u[-n-1] + (\frac{1}{3})^{n} u[n] \]

(iii) \( |z| > \frac{1}{2} \)

First term \( |z| > \frac{1}{3} \) & second term \( |z| > \frac{1}{3} \)

\[ X[n] = 3(\frac{1}{2})^{n} u[n] + (\frac{1}{3})^{n} u[n] \]
12.3 (b)

\[ X_a(z) = 1 + 2z^{-2} - 5z^{-3}, \quad |z| > 0 \]

\[ X_a[n] = ? \]


\[ X_b(z) = \sin z, \quad |z| > 0 \]

using taylor series, ...

\[ X_b(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \]

\[ X_b[n] = 5[n] + \frac{1}{3!} 5[n+3] + \frac{1}{5!} 5[n+5] - \frac{1}{7!} 5[n+7] + \ldots \]

\[ X_b[n] = \sum_{k=0}^{\infty} \frac{(-1)^k}{5[n+2k+1]} \]
12.3 (c)

\[ X(z) = \ln(1 + z^{-1}) \quad , \quad |z| > 1 \]

\[ x[n] = ? \]

**Differentiation property:**

\[
 n \times x[n] \longleftrightarrow -z^{-1} \frac{dX(z)}{dz}
\]

so

\[
 n \times x[n] = z^{-1} \left\{ -z^{-2} \frac{dX(z)}{dz} \right\}
\]

so

\[
 \frac{dX(z)}{dz} = \frac{1}{1 + z^{-1}} \cdot -z^{-2}
\]

\[
 -z \frac{dX(z)}{dz} = \frac{z^{-1}}{1 + z^{-1}}
\]

\[
 z^{-1} \left\{ -z^{-2} \frac{dX(z)}{dz} \right\} = z^{-1} \left\{ \frac{z^{-1}}{1 + z^{-1}} \cdot z^{-1} \right\} = (-1)^{n-1} u[n-1]
\]

so

\[
 n \times x[n] = (-1)^{n-1} u[n-1]
\]

\[
 x[n] = \frac{(-1)^n \cdot u[n-1]}{n}
\]

\[
 x[n] = \frac{(-1)^n}{n} u[n-1]
\]
12.4

(1) \( H(z) = \frac{z^2}{z-1} \), \(|z| > 1\)

so

- but is it causal?

\[
H(z) = \frac{1}{z-1} \cdot z^2
\]

using time shift property

\[
x[n]\rightarrow x[n-n_0]
\]

\[
h[n] = u[n+1]
\]

right sided signal but noncausal

(2) Which system function(s) can be both stable and causal?

(a) \( H_a(z) = \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} \)

\[h_a[n] = \delta[n-2]\]

causal & stable

- Converges for all \( z \) except \( 0 \to \) so both conditions are possible.

1. Causal if ROC is exterior of circle including \( \infty \).
2. Stable if ROC includes unit circle \( |z|=1 \).

(b) \( H_b(z) = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{5}{2}z + 1} \)

\[= \frac{2z^2 - \frac{5}{2}z}{(z-2)(z-\frac{1}{2})} \]

poles at \( |z| = 2 \) & \( |z| = \frac{1}{2} \)

no way to include \( |z| = 1 \)

and make the ROC the exterior of a circle

(c) \( H_c(z) = \frac{(1-e^{-j\frac{\pi}{4}}z^{-1})(1-e^{-j\frac{\pi}{4}}z^{-1})}{(1-0.98e^{-j\frac{\pi}{4}}z^{-1})(1-0.98e^{j\frac{\pi}{4}}z^{-1})} \)

poles at \( 0.98e^{-j\frac{\pi}{4}} = 0.98(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}) \)

\[= 0.98 \frac{\sqrt{2}}{2} + j 0.98 \frac{\sqrt{2}}{2} \]

\[= 0.693 + j 0.693 \]

so it is possible to have ROC exterior of circle & include unit circle

(d) \( H_d(z) = \frac{2 + \frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})z^{-1}} = \frac{2z + \frac{1}{2}}{z - \frac{1}{2}} \)

\[= \frac{(2z+1)}{(z-\frac{1}{2})} \]

ROC will not include \( \infty \)

because it extra zero

So \( H_a(z) \& H_c(z) \)
\[ H_{12}(z) = \frac{(z+1)(z-(\frac{1}{2}+j\frac{\sqrt{3}}{2}))(z-(\frac{1}{2}-j\frac{\sqrt{3}}{2}))(z-(-\frac{1}{2}+j\frac{\sqrt{3}}{2}))(z-(-\frac{1}{2}-j\frac{\sqrt{3}}{2}))}{z^5} \]

\[ H_{15}(z) = \frac{(z+1)(z^2-z+1)(z^2+z+1)}{z^5} = \frac{(z+1)(z^4+z^2+1)}{z^5} = \frac{z^5+z^4+z^3+z^2+z+1}{z^5} \]

\[ H_{2}(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} \]

Corresponds to \( h_d \in \mathbb{C}^3 \)

\[ H(e^{i\theta}) = \prod_{n=1}^{\mu} \frac{\text{length of vector connecting } i\text{th zero to } e^{i\theta}}{\prod_{n=1}^{\mu} \text{(length of vector connecting } j\text{th pole to } e^{i\theta})} \]

For \( w = 0 \): \[ \sqrt{(3^2 + (\frac{\sqrt{3}}{2})^2)} = \sqrt{13} \]

Length of zeros:

\[ (\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}i) \]

\[ \text{length} = \sqrt{(\frac{3}{2})^2 + (\frac{3}{2})^2} = 1 \]

\[ (\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}i) \]

\[ \text{length} = \sqrt{(\frac{3}{2})^2 + (-\frac{3}{2})^2} = 1 \]

Lengths of poles:

\[ e^{i\frac{\pi}{10}} \]

\[ \text{all 5 poles have length 1} \]

\[ |H(e^{i\theta})| = \frac{1 \cdot 1 \cdot \sqrt{3} \cdot \sqrt{3} \cdot 2}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1} = 6 \]

At \( \theta = \frac{\pi}{3} \times 1 \):

\[ \text{this zero will cause } |H(e^{i\theta})| \text{ to have zero in numerator} \]

\[ \frac{\pi}{10} \]

\[ \text{corresponds to } |H_{12}(e^{i\theta})| \]

\[ \text{corresponds to } |H_{15}(e^{i\theta})| \]
System II:

\[ h_{II}(z) = \frac{z - \frac{3}{4}}{z + \frac{3}{4}} = \frac{1}{1 + \frac{3}{4}z^{-1} - \frac{3}{4}z^{-2}} \]

For causal system:

\[ h_{II}[n] = (\frac{3}{4})^n u[n] - \frac{3}{4} (\frac{3}{4})^n u[n-1] \]

- Time shift property: \( x[n] \rightarrow x[n-m] \)

\[ h_{II}[n] = (\frac{3}{4})^n u[n] + (\frac{3}{4})^{n-1} u[n-1] \]

Corresponds to \( h_E[n] \)

- At \( w=0 \) \( \frac{1/4}{1/4} = \frac{1}{7} \)
- Increases until
- At \( w=\pi \) \( \frac{3/4}{1/4} = 7 \) corresponds to \( |H_2(e^{jw})| \)

System III:

\[ h_{III}(z) = \frac{z + \frac{3}{4}}{z - \frac{3}{4}} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{3}{4}z^{-2}} \]

For causal system:

\[ h_{III}[n] = (\frac{3}{4})^n u[n] + \frac{3}{4} (\frac{3}{4})^n u[n-1] \]

\[ h_{III}[n] = (\frac{3}{4})^n u[n] + (\frac{3}{4})^{n-1} u[n-1] \]

Corresponds to \( h_E[n] \)

- At \( w=0 \) \( \frac{3/4}{1/4} = \frac{7}{1} \)
- Decreases until
- At \( w=\pi \) \( \frac{1/4}{1/4} = \frac{1}{7} \) corresponds to \( |H_5(e^{jw})| \)

Note: On these two problems the magnitude plots start and end at around 9.

This means that the poles and zeros should probably be at \( \frac{1+x}{1-x} = 9 \Rightarrow x = \frac{4}{5} \)

So, not same shapes though.
System IV: so assuming poles at $\frac{1}{2} \pm \frac{i}{2}$ 4 zeros at $\frac{1}{2} \pm j\left(\frac{\sqrt{2}}{2}\right)$ say $\frac{1}{2} \pm \frac{1}{2}j$

$$H_{IV}(z) = \frac{(z - \left(\frac{1}{2} + j\frac{\sqrt{2}}{2}\right))(z - \left(\frac{1}{2} - j\frac{\sqrt{2}}{2}\right))}{(z - \left(\frac{1}{2} + j\frac{1}{2}\right))(z - \left(\frac{1}{2} - j\frac{1}{2}\right))} = \frac{z^2 - z + \frac{41}{64}}{z^2 - z + \frac{1}{2}}$$

To use partial fractions must use long division.

$$\frac{1}{z^2 - z + \frac{1}{2}} = \frac{\frac{9/64}{z - (\frac{1}{2} + j\frac{1}{2})}}{z^2 - z + \frac{1}{2}} + \frac{\frac{9/64}{z - (\frac{1}{2} - j\frac{1}{2})}}{z^2 - z + \frac{1}{2}}$$

so $H(z) = 1 + \frac{\frac{9/64}{z - (\frac{1}{2} + j\frac{1}{2})}}{z - (\frac{1}{2} - j\frac{1}{2})}$

$$A = \left.\frac{\frac{9/64}{(\frac{1}{2} + j\frac{1}{2})}}{z^2 - z + \frac{1}{2}}\right|_{z = \frac{1}{2} + j\frac{1}{2}} = \frac{9/64}{j} = -\frac{9/64}{j}$$

$$B = \left.\frac{\frac{9/64}{(\frac{1}{2} - j\frac{1}{2})}}{z^2 - z + \frac{1}{2}}\right|_{z = \frac{1}{2} - j\frac{1}{2}} = \frac{9/64}{j} = \frac{9/64}{j}$$

so $H(z) = 1 - \frac{9/64j}{z - (\frac{1}{2} + j\frac{1}{2})}z^{-1} + \frac{9/64j}{z - (\frac{1}{2} - j\frac{1}{2})}z^{-1}$

$$h[n] = 5[n] - \frac{9}{64j} \left(\frac{1}{2} + j\frac{1}{2}\right)^{n-1}u[n-1] + \frac{9}{64j} \left(\frac{1}{2} - j\frac{1}{2}\right)^{n-1}u[n-1]$$

Most of the time the lengths of the zeros are larger than the length of the poles.

Specifically between $0 < \omega < \pi$, the zeros have a little bigger $\pi$ poles result = a little smaller = a little bigger than 1.

For these omega's the lengths of the poles are larger than the lengths of the zeros.

$$at \omega = \frac{\pi}{2};$$

$\pi$ zero length = a little smaller $\pi$ pole length = a little bigger = a little smaller than 1.

Corresponds to $|H_{IV}(e^{j\omega})|$
System V: Looks like

\[ H_2(z) = \frac{z - 3/4}{z} = 1 - 3/4 z^{-1} \]

\[ h_0 C_n[j] = 5 C_n[j] - \frac{3}{4} \delta C_{n-1} \]

Corresponds to \( h_0 C_n[j] \)

at z = 0 \[ \frac{1}{4} = \frac{1}{4} \]

increases until at z = ±\( \frac{3}{4} \) \[ \frac{3}{4} = \frac{7}{4} = 1.75 \]

Corresponds to \( |H_1(e^{j\omega})| \)

System VI: Looks like

\[ H_{VI}(z) = \frac{z^2}{(z - \frac{3}{4}j)(z + \frac{3}{4}j)} = \frac{z^2}{z^2 - (\frac{3}{4})^2} = \frac{1}{z^2 + (\frac{3}{4})^2} \]

\[ s_t = \frac{z^2 + 16/25}{z^2 - 25/25} = \frac{1}{z^2 + (\frac{3}{4})^2} \]

so \( H_{VI}(z) = 1 - \frac{16/25}{(z - \frac{3}{4}j)(z + \frac{3}{4}j)} \)

\[ G_7(z) = \frac{16/25}{(z - \frac{3}{4}j)(z + \frac{3}{4}j)} = \frac{A}{z - \frac{3}{4}j} + \frac{B}{z + \frac{3}{4}j} \]

\[ A = (z - \frac{3}{4}j)b(z) \]
\[ b(z) = \frac{1}{z + \frac{3}{4}j} \]
\[ B = (z + \frac{3}{4}j)b(z) \]
\[ b(z) = \frac{1}{z - \frac{3}{4}j} \]

\[ h_{VI}(z) = 1 + \frac{3/4j z^{-1}}{1 - (3/4j) z^{-1}} - \frac{3/4j z^{-1}}{1 + (3/4j) z^{-1}} \]

\[ h_{VI} C_n[j] = 5 C_n[j] + \frac{3}{4} j C_n[j] - \frac{3}{4} j C_{n-1}[j] \]

\[ h_{VI} C_n[j] = 5 C_n[j] + \frac{3}{4} j C_{n+1}[j] + \frac{1}{2} (-\frac{3}{4}j)^{n-1} C_{n-1}[j] \]

\[ 0 \leq \omega \text{ for } n \text{ odd} \]

Corresponds to \( h_{VI} C_n[j] \)

Ratio of zeros to poles is highest at around z = ±\( \frac{3}{4} \)

\[ \frac{1.1}{(1/2)(1/4)} = \frac{1}{\sqrt{2}} = 2.77 \]

at \( \omega = 0 \) length of poles is larger than zeros

\[ \frac{1.1}{\sqrt{2}} \]

Corresponds to \( \sqrt{2} \)

\[ \frac{1.1}{\sqrt{2}} \]

\[ \text{Corresponds to } |H_3(e^{j\omega})| \]

\[ 12 \]