Announcements

- The final exam will be held on Thursday, 5/24 from 9 am to Noon in Johnson Ice Rink.
- Marathon OH are next Tuesday and Wednesday from 2-7 pm.
- Final exam reviews will be held on Thursday, 5/17 from 7:30 to 9:30 pm and on Friday, 5/18 from 2:00 to 4:00 pm. Both sessions will contain the identical contents and are in 32-123.

Today’s Agenda

- Introduction to the z-Transform
  - Eigenfunction Property
  - Definition of z-Transform
- Poles and Zeros
  - Caveats about poles at zero and infinity
- Region of Convergence and Properties
- The Inverse z-Transform
  - $z$ vs. $z^{-1}$
- The z-Transform and LTI Systems
1 Introduction to the \( z \)-Transform

1.1 Eigenfunction Property

In much the same way that the CT Laplace transform was derived from the eigenfunction property, we can similarly derive the DT \( z \)-transform. In lecture we showed that a certain set of input signals, namely complex exponentials of the form \( x[n] = z^n \) are eigenfunctions of LTI systems, meaning the corresponding outputs are simply scaled versions of the inputs of this form, and this scaling factor is the eigenvalue. Again, the outputs of DT LTI systems in response to \( x[n] = z^n \) are \( y[n] = H(z)z^n \) where the eigenvalues \( H(z) \) associated with the given eigenfunctions are:

\[
H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}.
\]

where \( h[n] \) is the impulse response of the DT LTI System.

Again, the main motivation for our development of the Z-transform in the first place is the eigenfunction property of LTI systems:

---

**The Eigenfunction Property of DT LTI Systems:**

If the input \( x[n] \) to a DT LTI system with transfer function \( H(z) \) is a complex exponential of the form:

\[
x[n] = z^n,
\]

where \( z \) is in the ROC of \( H(z) \), then the output \( y[n] \) is:

\[
y[n] = H(z)z^n.
\]

This is a very important concept in 6.003 which we will see time and time again.

---

1.2 Definition of \( z \)-Transform

From the general form of the eigenfunction property of DT LTI systems in the previous section, we can define the Z-transform as follows:

---

**The \( z \)-transform:**

The \( z \)-transform \( X(z) \) of a DT signal \( x[n] \) is:

\[
X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}.
\]
We write this as:

\[ X(z) = \mathcal{Z}\{x[n]\} \]

\[ x[n] \rightleftharpoons_X X(z) \]

The inverse \(z\)-transform is written as:

\[ x[n] = \mathcal{Z}^{-1}\{X(z)\}. \]

1.3 The relationship between the \(z\)- and Fourier transforms

We discovered a relationship between the Laplace and Fourier transforms in CT; let’s do the same in DT for the \(z\)- and Fourier transforms. Let’s write the variable \( z \) in polar form \( z = re^{j\omega} \), where \( r \) is the magnitude of \( z \) and \( \omega \) is the phase of \( z \). If we set \( r \) to 1, then the \( z\)-transform reduces to the Fourier transform:

\[
X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = \mathcal{F}\{x[n]\}. \]

Now if we plug in the polar form of \( z \) into the definition of the \(z\)-transform and recombine terms, we get the Fourier transform of \( x[n]r^{-n} \):

\[
X(z)\big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n}
= \sum_{n=-\infty}^{+\infty} (x[n]r^{-n}) e^{-j\omega n}
= \mathcal{F}\{x[n]r^{-n}\}. \]

To summarize:

<table>
<thead>
<tr>
<th>The Fourier and (z)-Transforms:</th>
</tr>
</thead>
</table>

The \(z\)-transform \(X(z)\) of a DT signal \(x[n]\) is the Fourier transform of \(x[n]r^{-n}\), where \(r\) is the magnitude of \(z\), and the Fourier transform \(X(e^{j\omega})\) is the \(z\)-transform evaluated at \(z = e^{j\omega}\).

The \(j\omega\)-axis played a special role in the Laplace transform; it was the location of the CT Fourier transform. We’ll see that the unit circle in the \(z\)-transform plays an analogous role; it is the location of the DT Fourier transform. Basic properties of those transforms are consistent with their location. For instance, the CT Fourier transform is generally aperiodic in \(\omega\), and so it occupies an infinitely long line of the \(s\)-plane. The DT Fourier transform is periodic with period \(2\pi\) in \(\omega\), so it occupies a unit circle in the \(z\)-plane, which has length \(2\pi\) to enforce the periodicity.
2 Poles and zeros

Many of the $z$-transforms we come across are ratios of polynomials:

$$X(z) = \frac{N(z)}{D(z)},$$

where $N(z)$ and $D(z)$ are the numerator and denominator polynomials, respectively. Rational $z$-transforms result from signals that are linear combinations of complex exponentials and from LTI systems described as linear constant-coefficient difference equations. Once again, we have the concept of poles and zeros of rational $z$-transforms, including multiple poles and zeros and poles and zeros at infinity and zero. However, these behave a bit differently than they do in the Laplace transform, as we’ll see later.

2.1 Caveats about poles at zero and infinity

We have to be careful about poles at zero and infinity for $z$-transforms, even more so than we did for Laplace transforms. For example, property 3 above states that a finite-duration signal might have poles at zero or infinity. From the definition of the $z$-transform, we see that if such a signal $x[n]$ has non-zero values for some $n < 0$, then its transform has terms with positive powers of $z$. Thus, it has a pole at infinity, and the ROC does not include $z = \infty$. Likewise, the transforms of a signal that has non-zero values for some $n > 0$ has negative powers of $z$, so it has a pole at zero and the ROC does not include $z = 0$. Thus, the only way for the ROC of a transform to be the entire $z$-plane, including at $z = 0$ and $z = \infty$ is for the corresponding signal $x[n]$ to be zero for all non-zero time, so $x[n]$ is of the form:

$$x[n] = A\delta[n],$$

where $A$ is a constant. The same reasoning applies for the last parts of properties 8 and 9.

We also saw with the Laplace transform properties that ROCs sometimes have the words “at least” attached, which comes about when there is pole-zero cancellation. $z$-transform properties have the same problem, but they are even worse because now we also need to watch out for the “except for the possible addition or deletion of the origin” for the ROC of a time-shifted signal. The next problem illustrates these issues.

3 Region of Convergence and Properties

Like the Laplace transform, the $z$-transform converges for only certain values of $z$, namely those $z$ such that $x[n]r^{-n}$ is absolutely summable, where $r$ is the magnitude of $z$. These values are in the region of convergence (ROC) of the $z$-transform.

For example, the right-sided exponential:

$$x[n] = a^n u[n]$$

has the following $z$-transform with the associated ROC:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|. $$
The left-sided exponential:

\[ x[n] = -a^n u[-n - 1] \]

has the same algebraic z-transform with a different ROC:

\[ X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|. \]

These signals are analogous to the CT single-sided exponentials with the same differences: Flipping the ROC is equivalent to time reversal and a minus sign. One annoying asymmetry is that although the CT case goes from \( u(t) \) to \( u(-t) \) when the ROC is flipped, the DT case goes from \( u[n] \) to \( u[-n - 1] \), rather than \( u[-n] \).

Like Laplace, we have:

**Specifying the Region of Convergence:**

When writing the z-transform of a DT signal, both the algebraic expression and the region of convergence are required.
The textbook outlines nine basic properties of the region of convergence of the $z$-transform:

<table>
<thead>
<tr>
<th>Properties of the Region of Convergence of the $z$-Transform:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The ROC consists of a ring in the $z$-plane centered about the origin.</td>
</tr>
<tr>
<td>2. For rational $z$-transforms, the ROC does not contain any poles.</td>
</tr>
<tr>
<td>3. If the signal is of finite duration, then the ROC is the entire $z$-plane, except possibly at $z = 0$ and/or $z = \infty$.</td>
</tr>
<tr>
<td>4. If the signal is right-sided, and if the circle $</td>
</tr>
<tr>
<td>5. If the signal is left-sided, and if the circle $</td>
</tr>
<tr>
<td>6. If the signal is two-sided, and if the circle $</td>
</tr>
<tr>
<td>7. If the $z$-transform is rational, then its ROC is bounded by poles or extends to infinity.</td>
</tr>
<tr>
<td>8. If the $z$-transform is rational and the signal is right-sided, then the ROC is the region of the $z$-plane outside the of the outermost pole. Furthermore, if the signal $x[n]$ is zero for $n &lt; 0$, then the ROC also includes $z = \infty$.</td>
</tr>
<tr>
<td>9. If the $z$-transform is rational and the signal is left-sided, then the ROC is the region of the $z$-plane inside the of the innermost pole. Furthermore, if the signal $x[n]$ is zero for $n &gt; 0$, then the ROC also includes $z = 0$.</td>
</tr>
</tbody>
</table>
Problem 12.1

Find the $z$-transforms and the associated ROCs of the following finite-length DT signals. Be careful about poles at zero and infinity.

(a) $x_a[n] = 3\delta[n + 4] - \delta[n + 3] + 2\delta[n + 2]$.

(b) $x_b[n] = x_a[n - 3] = 3\delta[n + 1] - \delta[n] + 2\delta[n - 1]$.

(c) $x_c[n] = 2^n u[n - 1] + 4^n u[-n]$.

(d) $x_d[n] = \left(\frac{1}{2}\right)^n u[-n + 4]$.

(Work space)
Problem 12.2

Find the $z$-transform $X(z)$ and the associated ROC of the signal $x[n]$, where $x[n]$ is a semi-periodic signal that satisfies the following conditions:

- $x[n] = 0$ for $n < 0$.
- $x[0] = 2$.
- $x[1] = 3$.
- $x[n] = x[n - 3]$ for $n > 2$.

There are several ways to solve this problem!

(Work space)
4 The Inverse $z$-Transform

As with Laplace, taking the inverse $z$-transform generally requires contour integration, but we can use partial fraction expansion and the table of properties. We will often do partial fraction expansion in $z^{-1}$ instead of $z$.

The partial fraction expansion for $z$-transform is calculated in the same way as Laplace (the partial fraction expansion is just an algebraic manipulation). Thus, we can treat the variable $z$ or $z^{-1}$ in the same way we treated $s$.

Of course, just like with Laplace, pattern-matching and using tables of properties is another favorite way of finding the inverse $z$-transforms.

4.1 $z$ vs. $z^{-1}$

We can use two different forms of rational $z$-transforms:

(a) Polynomials in $z$, or the products of factors of the form $(z - a)$.
(b) Polynomials in $z^{-1}$, or the products of factors of the form $(1 - az^{-1})$.

In both forms, the constant $a$ indicates the poles and zeros. It doesn’t really matter which form you use, just be consistent! For example, suppose we would like to find the inverse $z$-transform of the following:

$$X(z) = \frac{z^3}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}.$$ 

Three possible ways of doing this are:

- Find the inverse transform of:

  $$X_1(z) = z^{-3}X(z) = \frac{1}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})},$$

  then use the time-shift property. This is generally the easiest method.

- Rewrite the transform as:

  $$X(z) = \frac{z^5}{(z + \frac{1}{4})(z - \frac{1}{2})},$$

  and do long division in $z$, followed by partial fraction expansion in $z$.

- Rewrite the transform as:

  $$X(z) = \frac{1}{z^{-3}(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})},$$

  and do partial fraction expansion in $z^{-1}$. This is rather tedious because there are multiple roots.

Keep in mind when doing partial fraction expansion to use long division for top-heavy rationals (when the numerator is of the same or higher order than the denominator).
5 The $z$-Transform and LTI Systems

Just like we used the impulse response $h[n]$ to analyze LTI systems, we can do the same with the $z$-transform. Recall that:

$$y[n] = h[n] \ast x[n],$$

where $x[n]$ is the input of a DT LTI system, $h[n]$ is the impulse response, and $y[n]$ is the output. Then, by the convolution property of the $z$-transform, we have:

$$Y(z) = X(z)H(z),$$

where the uppercase functions in $z$ are the $z$-transforms of the corresponding lowercase functions in $n$. We call $H(z)$ the transfer function, or system function, of the DT LTI system. We now have two ways of characterizing DT LTI systems:

<table>
<thead>
<tr>
<th>Ways of Characterizing DT LTI Systems:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The impulse response $h[n]$.</td>
</tr>
<tr>
<td>2. The frequency response $H(e^{j\omega})$, which is the transfer function $H(z)$ evaluated on the unit circle.</td>
</tr>
<tr>
<td>3. The transfer function $H(z)$, which is the $z$-transform of the impulse response $h[n]$.</td>
</tr>
</tbody>
</table>

The utility and choice of each method depends on the system and the problem we want to solve.
Problem 12.3  This problem illustrates using partial fraction expansion, applying properties and reading tables, or applying a power series expansion to find inverse z-transforms.

(a) Suppose we are given the z-transform:

\[ X(z) = \frac{4z^2 + \frac{1}{2}z}{z^2 - \frac{1}{6}z - \frac{1}{6}} = \frac{4 + \frac{1}{2}z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{4 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}. \]

For each of the following ROC’s, determine the corresponding signal \( x[n] \)

(i) \(|z| < \frac{1}{3} \).
(ii) \( \frac{1}{3} < |z| < \frac{1}{2} \).
(iii) \(|z| > \frac{1}{2} \).

(b) Find the signals that correspond to the following z-transforms and their associated ROCs:

(i) \( X_1(z) = 1 + 2z^{-2} - 5z^{-3}, \quad |z| > 0. \)
(ii) \( X_2(z) = \sin z, \quad |z| \geq 0. \)

(c) The z-transform \( X(z) \) and the associated ROC of a DT signal \( x[n] \) is:

\[ X(z) = \ln(1 + z^{-1}), \quad |z| > 1. \]

Find \( x[n] \).

(Work space)
5.1 Causality

Analogously with Laplace, we have:

A DT LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.

The “including infinity” condition is important. In order for this to be met for rational system functions, we need an additional condition:

Causal Rational Systems and the ROC:

A DT LTI system with a rational transfer function is causal if and only if:
(a) the ROC is the exterior of a circle outside the outermost pole; and (b) the order of the numerator of the transfer function does not exceed that of the denominator.

The second condition is rather subtle, for this does not occur in the Laplace transform. Why do we need it? If the order of the numerator exceeds that of the denominator, then we can use long division to create terms with positive powers of $z$, which correspond to DT impulses in $n < 0$. The following problem illustrates this point.

5.2 Stability

Recall that a DT system is stable if and only if the impulse response is absolutely summable. In that case, the frequency response, which is the transfer function evaluated on the unit circle, exists. Thus:

Stable Systems and the ROC:

An DT LTI system is stable if and only if the ROC of its transfer function $H(z)$ includes the unit circle, $|z| = 1$.

It follows that:

Rational Causal and Stable Systems and the ROC:

An DT LTI system described by a rational transfer function $H(z)$ is causal and stable if and only if all of the poles of $H(z)$ lie inside the unit circle.
Problem 12.4

(1) Find the impulse response $h[n]$ of the system with the following transfer function and verify that the system is non-causal, even though $h[n]$ is right-sided:

$$H(z) = \frac{z^2}{z-1}, |z| > 1.$$ 

(2) Which system function(s) can be both stable and causal?

(a) $H_a(z) = \frac{1}{z^2}$.

(b) $H_b(z) = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{8}{2}z + 1}$.

(c) $H_c(z) = \frac{(1 - e^{j\pi/4}z^{-1})(1 - e^{-j\pi}z^{-1})}{(1 - 0.98e^{j\pi/4}z^{-1})(1 - 0.98e^{-j\pi}z^{-1})}$.

(d) $H_d(z) = \frac{2 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})z^{-1}}$. 

75
**Geometric Evaluation of Rational $z$-Transforms**  As we did for Laplace transforms, lengths and angles of vectors connecting a point of interest to poles or zeros can be used to evaluate geometrically rational $z$-transforms. Suppose $H(z)$ is given as the system function. The frequency response, $H(e^{j\omega})$ is $H(z)$ evaluated along the unit circle in the $z-$ plane, i.e., $z = e^{j\omega}$ for $0 \leq \omega < 2\pi$ or $-\pi \leq \omega < \pi$ (again assuming that the ROC contains the unit circle). First, we look at the magnitude plot. In general, for a fixed $\omega$ we can think of $|H(e^{j\omega})|$ as

$$|H(e^{j\omega})| = \frac{\prod_{i=1}^{\# \text{ of zeros}} \text{ length of a vector connecting } i^{th} \text{ zero to } e^{j\omega}}{\prod_{j=1}^{\# \text{ of poles}} \text{ length of a vector connecting } j^{th} \text{ pole to } e^{j\omega}}.$$  

Here it is assumed that the constant gain term is unity. If $\#$ of zeros or poles is zero, then we define the product above to be 1. In our case, there are two poles and 1 zero. Thus, the above expression can be simplified to:

$$|H(e^{j\omega})| = \frac{|v_3|}{|v_1||v_2|},$$

where $v_1, v_2, v_3$ are vectors shown in the figure below:

From the plot, we see that when $\omega = 0$, $|v_3|$ becomes maximum. Thus, we expect to have the maximum magnitude at the frequency. Also, when $\omega = \pi$, the magnitude becomes 0 because the zero is in the same location as $\omega$; thus the magnitude of $H(e^{j\omega})$ becomes minimum. The magnitude plot of $H(e^{j\omega})$ for $0 \leq \omega < \pi$ is shown below:
The phase $\angle H(e^{j\omega})$ can be described as:

$$\angle H(e^{j\omega}) = \sum_{i=1}^{\# \text{ of zeros}} \left( \text{angle of vector connecting } i \text{ th zero to } e^{j\omega} \right) - \sum_{j=1}^{\# \text{ of poles}} \left( \text{angle of vector connecting } j \text{ th pole to } e^{j\omega} \right).$$

For this example,

$$\angle H(e^{j\omega}) = \angle v_3 - (\angle v_1 + \angle v_2),$$

The phase starts off at 0 when $\omega = 0$ and decreases to $-\frac{3}{2}\pi$ when $\omega = \pi$. The phase plot is shown below:
Consider 6 causal real DT LTI systems with the corresponding pole-zero diagrams, impulse responses, and frequency response magnitudes (on 3 separate pages). Find the one-to-one-to-one correspondence between the three types of diagrams.
Problem 12.6

Suppose a DT LTI system has the following transfer function:

\[ H(z) = \frac{1}{z^3 + 2\sqrt{2}z^{3/2} + \frac{1}{8}}. \]

It is also known that the circle 

\[ |z| = \frac{1}{2} \]

is in the ROC of \( H(z) \). Find the output \( y[n] \) of the system when the input \( x[n] \) is:

\[ x[n] = \left( \frac{1}{2} \right)^n \sin \left( \frac{\pi}{3} n \right). \]

(Caution: Does \( x[n] \) have a \( z \)-transform?)

(Work space)