Omni-channel revenue management through integrated pricing and fulfillment planning

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Due to the rapid growth of e-commerce sales, many retailers have been shifting to omni-channel operations by integrating their network of brick-and-mortar stores with their nascent online channel. However, existing analytics infrastructure and models have not been able to keep pace with this development. We identify and address new omni-channel operational challenges: (i) in-store inventory is a shared resource across channels and locations due to cross-channel fulfillment such as ship-from-store and buy-online-pickup-in-store, and (ii) channel prices affect customers’ channel preferences. We model cross-channel customer interactions at the location level using attraction demand models. We then propose a tractable optimization model, which maximizes omni-channel profit using two controls: (i) cross-channel fulfillment inventories, and (ii) lifecycle channel prices across the retail network. We demonstrate using closed-form solutions and examples that if the e-commerce warehouse is capacitated, then compared to a price-control model, the dual-control model reduces price dispersion across the network and increases total sales and profits. We apply our approach for clearance pricing of 195 electronics SKUs of a major U.S. retailer. Our experiments show that compared to the retailer’s actual sales data, the omni-channel model results in an average of 7% increase in sales, a 6-12% increase in markdown revenue, and faster delivery times. Working with IBM Commerce, a proprietary implementation of the model is being piloted with the retailer to assess its performance.

Key words: Omni-channel, pricing, attraction demand, markdown pricing, e-commerce fulfillment, cross-channel, elasticity

1. Introduction

Omni-channel retailing is a recent trend sweeping companies across the retailing industry (Brynjolfsson et al. 2013, Bell et al. 2014). An omni-channel strategy promises to revolutionize how companies engage with customers by creating a seamless customer shopping experience through an alignment of the retailer’s multiple channels. The following are just a few examples of the new capabilities enabled by omni-channel retail. A customer might purchase a product from the online store while she is in a brick-and-mortar store after finding that the online price is cheaper through
browsing the retailer’s website in her mobile phone. A customer who purchased a product online might choose a “buy online, pick up in store” option since he prefers the convenience of receiving the product sooner than having to wait for the package to be shipped to his address. The package that an e-commerce customer receives might have been fulfilled from a nearby brick-and-mortar store since the e-commerce warehouse is out of stock.

From a customer’s perspective, an omni-channel environment makes it easier to compare prices from different channels (in-store and online), to purchase a product from any channel, and to receive the product conveniently through any of the retailer’s multiple cross-channel fulfillment options. These new conveniences allows the retailer to remain competitive in an increasingly competitive retail landscape. As e-commerce sales are expected to grow at an annual rate of 9.5% through 2018 (Forrester Consulting 2014b), an integration of multiple sales channels also allows an omni-channel retailer to efficiently support the growth in e-commerce sales through its network of brick-and-mortar stores. Unlike pure e-commerce retailers such as Amazon.com that have multiple e-commerce fulfillment centers, a cross-channel fulfillment network is a crucial advantage of an omni-channel retailer due to the prohibitively large investment cost per e-commerce warehouse. With cross-channel fulfillment, lost e-commerce sales can be avoided even if the e-commerce warehouse is out of stock. Additional benefits for cross-channel fulfillment include faster delivery times and the ability to direct e-commerce sales to stores with slow moving inventory. An operations manager for a books/media retailer stated that enabling stores to fulfill e-commerce sales has driven cost down by 18% and revenue up by 20% (Forrester Consulting 2014a).

Despite the many benefits of an omni-channel environment, it introduces many new operational challenges to the retailer on both the supply and demand side. On the supply side, inventory that has been traditionally dedicated to one channel (e.g. brick-and-mortar store inventory) can now be used to fulfill customer purchases from another channel (e.g. ship-from-store fulfillment, buy-online-pick-up-in-store). The new challenge is to determine the optimal use of the shared inventories based on the channel demands and channel profitability, both of which are affected by the omni-channel prices. On the demand side, the ease with which customers can purchase through any channel creates a new complication in setting the e-commerce price and the brick-and-mortar prices, since channel prices can affect a customer’s channel preferences. These price and inventory cross-channel interactions described here for an omni-channel retailer are significantly more complex compared to retailers that operate their channels independently.

Despite these challenges specific to the omni-channel environment, many omni-channel retailers set prices based on legacy price optimization systems developed for a single channel. In the past

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1 For example, Macy’s built a fulfillment center which was projected to cost $170 million (Rueter 2013).
few decades, price optimization has been widely embraced by retailers through popular pricing analytics solution providers. However, the state-of-the-art price optimization models determine prices independently for multiple channels, failing to account for any channel interdependencies or price-matching strategies between channels.

Utilizing legacy pricing systems in omni-channel retail can have negative consequences, which we highlight using the following real-world example from a major U.S. retailer selling a Tablet computer over a 12 week clearance period. Fig. 1 shows the channel prices (e-commerce price and average brick-and-mortar price) as well as total channel sales for a Tablet SKU that has been marked for clearance on Week 40. By the start of the clearance period, the retailer has no e-commerce warehouse inventory, and yet it continues to sell the item on the e-commerce portal. At the start of the clearance period, the legacy pricing software sets steep initial markdowns in the brick-and-mortar stores to clear off all store inventory in the next 12 weeks from walk-in store customers (see topmost panel). However, not all store inventory is at risk of becoming unsold since all e-commerce sales are fulfilled through ship-from-store during the clearance period (see middle panel). Because of the steep store markdowns, after an initial sales spike, total brick sales steadily decline due to an increasing number of stores stocking out (see bottom panel). In contrast to the steep brick markdowns, the initial e-commerce clearance price set by the legacy system is the regular price since there is no inventory tagged in the warehouse specifically for e-commerce sales. Aware of the significant channel price difference from the system, the retailer’s merchandise managers make ad-hoc adjustments using spreadsheets (override recommended prices, inject artificial e-fulfillment inventory) to bring the e-commerce price closer to the brick-and-mortar prices set by the system.

Figure 1 Time series data for the channel prices and the channel sales of a Tablet SKUs.
(see topmost panel). In discussions with merchandise managers, this manual adjustment process was revealed to be unmanageable, and time consuming, resulting in increased labor cost.

In the above example, setting the online price significantly higher than the brick price can cause channel cannibalization; setting it to be equal to the brick price can result in lower profitability due to channel-loyal customer segments or regional differences. This trade-off between channel profitability and channel cannibalization has to be carefully balanced in an omni-channel pricing strategy, which as noted above, also depends on the cross-channel fulfillment plan.

As part of a joint partnership with IBM Commerce, a leading provider of merchandizing solutions, we engaged with a major omni-channel retailer that was facing the operational challenges described above. Through this engagement, we developed an advanced omni-channel model for retail analytics. In this paper, we describe the proposed model, discuss key insights, and present a business value assessment through computational experiments with data from the retailer.

1.1. Contributions

The following are the contributions of our paper:

1. **Omni-channel pricing and cross channel fulfillment model (OCP-X):** For a short-lifecycle item without inventory replenishments, we propose the OCP-X optimization model that maximizes omni-channel profit through two controls: (i) cross-channel fulfillment inventories, and (ii) lifecycle channel prices across the omni-channel network. The OCP-X model is practically motivated with business constraints and several extensions including warehouse-to-store inventory replenishment. We use attraction demand models to characterize consumers’ channel choices by location and note that the resulting optimization model is a non-linear and non-convex problem. Using a sequence of mathematical transformations, the OCP-X model can be reformulated as a tractable mixed integer program (MIP) that can be solved efficiently through commercial off-the-shelf MIP solvers such as IBM ILOG CPLEX. The key benefits of the OCP-X model are in (a) directing e-commerce sales away stores that are likely to sell-out towards stores that have slow moving inventories, and (b) better managing channel demands using omni-channel prices.

2. **Special cases, closed-form solutions and the importance of cross-channel fulfillment:** Using special cases of the OCP-X model, we provide insight into the structure of optimal channel prices, as well as the benefits of joint price and cross-channel fulfillment controls. Specifically, for a single location scenario, we find closed form solutions, and show that the maximum deviation between the optimal channel prices is proportional to the cross-channel fulfillment cost. Moreover, if an omni-channel retailer can use both price and fulfillment controls to manage its tight capacity constraint on an e-commerce dedicated facility, then the “regret” is bounded,
which is not the case if the retailer just uses price controls. We define the regret as the difference between the unconstrained optimal profit minus the optimal profit with capacity constraints. In a two-location case, we also computationally show that price deviation across brick locations is significantly reduced by employing cross-channel fulfillment, also resulting in smaller regrets.

3. **Implementation and business value assessment:** We demonstrate the potential financial benefits of OCP-X through extensive computational experiments on three electronic product categories of a major retailer. We find that OCP-X results in an average of 7% sales increase and 6-12% revenue increase during the 10-12 week clearance period of 195 SKUs in these categories. Furthermore, our model also achieves a more than a 21% reduction in projected brick store lost sales and more than a 23% reduction in unsold inventory, while also improving delivery times on average. Working with IBM Commerce, a proprietary implementation of the OCP-X model is being piloted with this retailer to assess its performance, and a full-scale commercialization is expected.

2. **Literature review**

Omni-channel retailing is a relatively new area, there are only a few academic papers that are dedicated to the operations of omni-channel retail, including pricing and/or cross-channel fulfillment. We next discuss previous research that is relevant to our paper.

One relevant research stream is the literature on dynamic pricing in the retail industry, where the general model considers a limited set of perishable items being sold to price-sensitive customers over a finite horizon. For comprehensive surveys on pricing in revenue management, see Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003), Chen and Simchi-Levi (2012). Many of these models consider fashion items which are characterized by a short sales period and are ordered well in advance. Hence, these models often have only dynamic pricing controls without inventory replenishment. Gallego and van Ryzin (1994) assume demand follows a time-invariant Poisson process with an intensity nonincreasing in price. Bitran and Mondschein (1997) considers a model where the Poisson process intensity changes over time. Feng and Gallego (1995) study the problem of determining the optimal timing of a markdown or markup, given that price can only change once over the sales season. A paper by Su (2007) studies a clearance pricing problem where customers are strategic. Caro and Gallien (2012) study the development and implementation of a clearance pricing model for the fast-fashion retailer, Zara. Several papers also consider joint pricing and inventory control. Smith and Achabal (1998) assumes that demand depends not only on price but also on initial inventory. Bitran et al. (1998) generalize the single-store pricing problem to a retail chain that coordinates pricing across multiple stores and inventory redistribution is allowed between stores. Federgruen and Heching (1999) studies a model where a replenishment order may
be placed at the beginning of some periods. In these dynamic pricing models a common assumption is that customers can only purchase through one channel, which is not the case in omni-channel where customers have multiple purchase options.

Another related stream of research is dynamic pricing of multiple products with substitution, where customers can be satisfied with one of several options. In our work, substitution occurs among multiple sales channels. Pricing models with substitution is often in the context of static substitution, where customers’ choice does not depend on inventory availability and it is lost if the item is out-of-stock. Aydin and Porteus (2008) consider a joint pricing and inventory problem under price-based substitution using a single-period newsvendor type model. They assume a general class of demand models, under which the multinomial logit (MNL) and multiplicative competitive interactions (MCI) are special cases. Aydin and Porteus (2008), Akçay et al. (2010) show that under a MNL demand model, the profit is unimodal in the price space. Li and Huh (2011) show that the profit function is concave in the market share space under the more general nested logit demand models. Song and Xue (2007) characterize the optimal policy in a dynamic inventory and pricing model over a finite horizon assuming a general stochastic demand model which includes the MNL model. Dong et al. (2009) studies pricing under dynamic substitution (where customers’ choice is only among items in-stock) for multiple products with an MNL demand model when the customer’s choice is affected by price, quality and availability.

Also related is the literature related to dynamic pricing for multiple products with shared resources (Gallego and van Ryzin 1997, Maglaras and Meissner 2006). In omni-channel retail, channel inventories are shared resources due to the cross-channel fulfillment capability. Gallego and van Ryzin (1997) assume stochastic (Poisson) demand and study the pricing problem for the multi-product problem. They provide Hamilton-Jacobi-Bellman sufficient conditions for the problem, for which the closed-form solution is not available in general. Instead, they propose two simple heuristics based on the solution of the deterministic counterpart. The omni-channel problem we consider is related to this stream due to the shared inventories across channels. A common technique used in these models is to convert the pricing problem into an equivalent convex problem with the market shares (or demand rates) as controls. For example, we refer to Schön (2010), Keller et al. (2014) for efficient formulations under attraction demand models. In contrast, for the omni-channel problem e-commerce prices across locations are constrained to be equal, this makes the problem non-convex in the market share space.

Finally, we mention several recent papers on e-commerce and omni-channel retailing in operations and management science literature. Many papers study cross-channel distribution networks for e-commerce fulfillment (see surveys by Swaminathan and Tayur 2003, Agatz et al. 2008). Seifert et al. (2006) studies coordination contracts for an integrated supply chain where excess inventory at
retail stores can be used to fulfill e-commerce sales. Bendoly et al. (2007) study different models for integrating cross-channel inventories for fulfillment of e-commerce sales. However, these papers only consider the distribution network assuming prices are exogenous. With today’s modern systems, there is an opportunity to dynamically use prices to affect demand in different sales channels. Brynjolfsson et al. (2009) find in an empirical study that cross-channel competition between internet retailers and brick-and-mortar retailers is significant for mainstream products. A recent paper by Harsha et al. (2015) also finds significant cross-channel price elasticities for several office supply products. The thrust of their paper is for non-seasonal or long life cycle products where inventory can be planned in advance, and there is less risk of lost sales. They develop an omni-channel pricing model with static channel substitution and do not consider inventory availability and the possibility of cross-channel fulfillment in the retail chain.

3. The omni-channel demand model

The goal of an omni-channel retail strategy is to create a seamless customer shopping experience. One effect of this is that customers can easily compare prices between channels and choose to purchase from the one which gives the highest utility. Therefore, an omni-channel model should include cross-channel causals aside from the conventional same-channel causals. Examples of cross-channel causals are e-commerce price on a brick-and-mortar store demand or the brick-and-mortar store prices on the e-commerce demand. Since the brick-and-mortar channel consists of multiple stores, then the e-commerce demand is influenced by, not only the e-commerce price, but also the prices at all the brick-and-mortar stores.

The dependence of the e-commerce demand on the vector of all brick-and-mortar store prices presents a difficulty in parameter estimation of the demand model, because the number of parameters to be estimated is proportional to the number of brick-and-mortar stores. To make the demand estimation more tractable, we reduce the dimensionality of causal parameters by using the technique of Harsha et al. (2015) which introduces geospatial “zones” to tag e-commerce sales data and brick-and-mortar sales data. We assume that the channel choice of customers in a zone is independent of the causal factors in a different zone such as the local weather, brick-and-mortar prices and other brick competitors’ prices. Therefore, with respect to price causality alone, the zone-level channel demand only depends on two prices: the e-commerce price and the zonal brick-and-mortar price. Mathematically, the week $t$ channel demands in zone $z$ is a function of the week $t$ channel prices in zone $z$:

$$D_t^z(p^z) = \begin{pmatrix} D_t^z(p_{bz}^t, p_{cz}^t) \\ D_t^z(p_{bz}^t, p_{cz}^t) \end{pmatrix},$$

(3.1)
where $p^t_{bz}$ is the week $t$ brick-and-mortar channel price at zone $z$, $p^t_{ez}$ is the week $t$ e-commerce price at zone $z$, and $p^t_z = (p^t_{bz}, p^t_{ez})^T$.

Fig. 2 shows an example of the continental U.S. divided into 50 zones. The figure also shows the zonal distribution of sales (the volume is proportional to the pie size) and its split between channels (brick-and-mortar and e-commerce) for a product category of a major U.S. retailer. Note the heterogeneity of the e-commerce channel share across zones, which ranges from 4% to 11%.

3.1. Discrete choice model

We model the customer’s problem of choosing a purchase channel as a discrete choice model. These models are widely used in marketing research, economics and more recently in the revenue management literature to model a consumer’s substitutive choice.

A prominent discrete choice model is the multinomial logit (MNL) model. Suppose an individual in zone $z$ is considering purchasing a product. The individual obtains a utility for choosing a purchase channel. Let us denote by $u_{bz}$ his utility from purchasing from the brick-and-mortar channel, and $u_{ez}$ his utility from purchasing from the e-commerce channel. In particular, we assume

$$u_{bz} = \alpha_{bz} - \beta_{bz} p_{bz} + \gamma_{bz}^T \pi_{bz} + \xi_{bz},$$

$$u_{ez} = \alpha_{ez} - \beta_{ez} p_{ez} + \gamma_{ez}^T \pi_{ez} + \xi_{ez},$$

where $v_{mz}$ is the deterministic component and $\xi_{mz}$ is the stochastic error component of utility from choosing purchase channel $m \in \{b, e\}$. Note that $\beta_{mz}$ is the (negative) effect of channel price $p_{mz}$.
on utility, $\gamma_{mz}$ is the vector of effects of independent factors $\pi_{mz}$ on utility, and $\alpha_{mz}$ is the average effect of factors not included in the model. Some examples of factors that can be included in $\pi_{mz}$ are holiday effects, seasonality, and competitor prices. We assume that there is a “no purchase” option which gives zero utility. If we assume that the stochastic error components across individuals are distributed i.i.d. type I extreme value, then the customers probability of choosing channel $m$ takes the MNL form:

$$P_{mz} = \frac{\exp(v_{mz})}{1 + \exp(v_{mz}) + \exp(v_{m'z})}.$$  

Hence, suppose that $n^t_z$ denotes the market size for a product in zone $z$ at week $t$. Then the expected channel demands in zone $z$ are functions of channel prices at time $t$:

$$D^t_{bz} (p^t_z) = n^t_z \times \frac{\exp(a^t_{bz} - \beta_{bz}p^t_{bz})}{1 + \exp(a^t_{bz} - \beta_{bz}p^t_{bz}) + \exp(a^t_{ez} - \beta_{ez}p^t_{ez})},$$  

$$D^t_{ez} (p^t_z) = n^t_z \times \frac{\exp(a^t_{ez} - \beta_{ez}p^t_{ez})}{1 + \exp(a^t_{bz} - \beta_{bz}p^t_{bz}) + \exp(a^t_{ez} - \beta_{ez}p^t_{ez})},$$

where $a^t_{bz} = \alpha_{bz} + \gamma^T_{bz} \pi^t_{bz}$ and $a^t_{ez} = \alpha_{ez} + \gamma^T_{ez} \pi^t_{ez}$ are the non-price factor effects on utility.

A generalization of discrete choice models are the class of attraction demand models:

$$D^t_{bz} (p^t_z) = n^t_z \times \frac{f^t_{bz} (p^t_{bz})}{1 + f^t_{bz} (p^t_{bz}) + f^t_{ez} (p^t_{ez})},$$  

$$D^t_{ez} (p^t_z) = n^t_z \times \frac{f^t_{ez} (p^t_{ez})}{1 + f^t_{bz} (p^t_{bz}) + f^t_{ez} (p^t_{ez})}.$$  

The attraction function $f_i(\cdot)$, for $i \in \{b, e\}$, is a positive and strictly decreasing function of price. The attraction model has been used successfully in estimating demand in econometric studies, in marketing, and in operations. Special cases of the attraction model are the MNL model, the Multiplicative Competitive Interaction (MCI) model, and the linear attraction model.

4. **Model for the omni-channel integrated pricing and cross-channel fulfillment planning (OCP-X) problem**

In this section, we present a general model for optimizing channel prices and cross-channel inventories of an omni-channel retailer selling finite inventory over a finite horizon. The model we present can be reformulated as a mixed integer linear program which can be solved tractably using commercial off-the-shelf MIP solvers such as IBM ILOG CPLEX.

4.1. The retailer’s optimization model

Consider an omni-channel fulfillment network divided into $Z$ zones, where we assume one brick-and-mortar store per zone. The retailer sells a finite amount of inventory over $T$ consecutive weeks. For simplicity, we assume that the retailer has one e-fulfillment center (EFC) and one store per zone,
Figure 3  Omni-channel time-space network with two zones. Each zone has a brick inventory node, an e-commerce inventory node, a brick demand node, and an e-commerce demand node. Labels on arcs are flow quantities. Outflow arcs from the demand nodes have arc capacities equal to the demand. Dashed arcs are cross-channel fulfillment flows.

since the model can be trivially extended to the case with multiple e-commerce warehouse locations or multiple stores per zone. Let $x_e$ be the amount of inventory in the e-commerce warehouse, and $x_{bz}$ be the amount of in-store inventory in brick-and-mortar zone $z \in Z$.

During week $t$, the retailer chooses the e-commerce price $p^t_e$ to be offered in all zones, and the set of brick-and-mortar prices $\{p^t_{bz}\}_{z \in Z}$ to be offered in stores. After setting the price, the channel demands across zones $z \in Z$ are realized according to the omni-channel demand functions $D^t_{ez}(p^t_{ez}, p^t_e)$ and $D^t_{bz}(p^t_{bz}, p^t_e)$. The retailer then fulfills the demand using the available inventory in the fulfillment network. In our model, we assume that zone $z$ store demand can only be fulfilled by zone $z$ in-store inventory. However, e-commerce zone-level demand can be fulfilled either from the central e-commerce warehouse or from the in-store inventory of any zone in the fulfillment network. The latter option models ship-from-store fulfillment programs offered by many omni-channel retailers which allows store inventory to be shipped directly to any e-commerce customer.

Fig. 3 is a time-space network that shows demand-supply matching in the cross-channel fulfillment network. Each zone has a brick inventory node, an e-commerce inventory node, a brick demand node, and an e-commerce demand node. Inventory flows along directed arcs connecting inventory nodes to demand nodes. The dashed arcs are cross-channel fulfillment flows. Inter-zone flows connect the e-commerce inventory (EFC) to zonal e-commerce demand, and inter-channel
flows connect the brick inventory node to the e-commerce demand node in any zone. The outflow from the demand nodes are bounded by an arc capacity equal to the demand (a function of zone prices). Hence, not only are zones coupled by the constraint that e-commerce price is the same across all zones, but as the figure shows, zones are also coupled by inventory flow constraints.

After fulfilling demand, inventory in the network is depleted and the retailer incurs the corresponding fulfillment cost. We denote by $S_{tbz}^t$ the total fulfilled brick-and-mortar demand in zone $z$, which we assume to incur zero fulfillment cost. We denote by $S_{te,e(z,z)}^t$ the e-commerce demand at zone $z$ that has been fulfilled from the central e-commerce warehouse, which we assume to incur a per-unit fulfillment cost $c_{e(e,z)}$. We denote by $S_{te,e(z',z)}^t$ the e-commerce demand at zone $z$ that has been fulfilled using in-store inventory from zone $z'$, which we assume to incur a per-unit fulfillment cost $c_{e(z',z)}$. We denote by $S_{ez}^t = S_{te,e(z,z)}^t + \sum_{z' \in Z} S_{te,e(z',z)}^t$ the total fulfilled e-commerce demand originating from zone $z$. After the final week $T$, the leftover inventory in the central warehouse and in the brick-and-mortar stores can be salvaged at a per-unit value $q$.

The retailer’s omni-channel integrated price optimization and cross-channel fulfillment planning problem is given in Fig. 4 and denoted by OCP-X. The list of notation used in the optimization problem and their definitions can be found in Table 1. The decision variables are the channel prices by week-zone, the channel sales variables by week-zone, the brick-and-mortar store sales variables by week, the e-commerce sales by week-zone-fulfillment, and the leftover inventory in the EFC and in each brick-and-mortar store. The extensions of the omni-channel optimization model will be discussed at the end of this section.

The objective function (4.1a) of the OCP-X model is the retailer’s total profit over all weeks, all channels, and all zones. The first part of the objective is the total sales revenue over all weeks and

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**Table 1** Notation for the OCP-X model.

<table>
<thead>
<tr>
<th>Indices</th>
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<tbody>
<tr>
<td>$z, t$</td>
<td>a zone, a week</td>
</tr>
<tr>
<td>$e, b$</td>
<td>the e-commerce channel, the brick-and-mortar channel,</td>
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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$Z, T$</td>
<td>all zones, all weeks</td>
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<tr>
<td>$\Omega$</td>
<td>all feasible channel-zone price trajectories,</td>
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<table>
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<tr>
<th>Parameters</th>
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<tbody>
<tr>
<td>$x_{bz}$</td>
<td>initial inventory at store in zone $z$, initial inventory at EFC,</td>
</tr>
<tr>
<td>$c_{e(z,z)}$</td>
<td>per-unit cost from fulfilling an e-commerce order by zone $z$ customers using EFC inventory,</td>
</tr>
<tr>
<td>$c_{e(z',z)}$</td>
<td>per-unit cost from fulfilling an e-commerce order by zone $z$ customers using store inventory from zone $z'$,</td>
</tr>
<tr>
<td>$q$</td>
<td>per-unit salvage value for inventory leftover after the final week,</td>
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<tr>
<th>Decision Variables</th>
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<tr>
<td>$P_{tbz}^t$</td>
<td>brick-and-mortar price at zone $z$, week $t$,</td>
</tr>
<tr>
<td>$P_{te}^t$</td>
<td>e-commerce price at week $t$,</td>
</tr>
<tr>
<td>$S_{tbz}^t$</td>
<td>fulfilled brick-and-mortar, e-commerce demand from zone $z$ on week $t$,</td>
</tr>
<tr>
<td>$S_{te,e(z,z)}^t$</td>
<td>fulfilled e-commerce demand from zone $z$ on week $t$ using EFC inventory,</td>
</tr>
<tr>
<td>$S_{te,e(z',z)}^t$</td>
<td>fulfilled e-commerce demand from zone $z$ on week $t$ using store inventory from zone $z'$,</td>
</tr>
<tr>
<td>$Y_{e}$</td>
<td>leftover inventory in the EFC after the final week,</td>
</tr>
<tr>
<td>$Y_{bz}$</td>
<td>leftover inventory in brick-and-mortar store in zone $z$ after the final week.</td>
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</table>
Figure 4 The omni-channel integrated price optimization and the cross-channel fulfillment planning model.

OCP-X: maximize \[ \sum_{t \in T} \sum_{z \in Z} (P_{tz}^s S_{t}^z + P_{t}^e S_{t}^e) + q \left( Y_e + \sum_{z \in Z} Y_{bz} \right) - \sum_{t \in T} \sum_{z \in Z} \left( c_{(e,z)} S_{t}^{e,(e,z)} + \sum_{z' \in Z} c_{(z',z)} S_{t}^{e,(z',z)} \right), \]

subject to \[
\begin{align*}
S_{t}^z & \leq D_{t}^z (P_{tz}^s, P_{t}^e), & \forall z & \in Z, \forall t & \in T, \tag{4.1a} \\
S_{t}^e & \leq D_{t}^e (P_{tz}^s, P_{t}^e), & \forall z & \in Z, \forall t & \in T, \tag{4.1b} \\
\sum_{t \in T} \left( S_{t}^z + \sum_{z' \in Z} S_{t}^{e,(z,z')} \right) + Y_{bz} & = x_{bz}, & \forall z & \in Z, \tag{4.1c} \\
\sum_{t \in T} \sum_{z \in Z} S_{t}^{e,(z,z')} + Y_{e} & = x_{e}, \tag{4.1d} \\
S_{t}^{e,(z,z')} & = \sum_{z' \in Z} S_{t}^{e,(z,z')} + \sum_{z' \in Z} S_{t}^{e,(z',z)}, & \forall z & \in Z, \forall t & \in T, \tag{4.1e} \\
(P_{tz}^s, P_{t}^e)_{t \in T, z \in Z} & \in \Omega, \tag{4.1f} \\
S_{t}^z, S_{t}^z, S_{t}^{e,(z,z')}, S_{t}^{e,(z',z)} & \geq 0, & \forall z, z' & \in Z, t & \in T. \tag{4.1g} \\
Y_e, Y_{bz} & \geq 0, & \forall z & \in Z. \tag{4.1h}
\end{align*}
\]

all zones. In clearance pricing, leftover inventory after the final week is assumed to net a salvage value. Hence, a second term in the objective function is the total salvage value. The negative part of the objective is the total cost-to-fulfill e-commerce orders using the EFC and ship-from-store inventory. Fulfillment costs are important in the omni-channel planning model because, with everything being equal between two locations, this cost is used to break ties so that the item it chosen to be fulfilled from a closer location. Without loss of generality, we assume that there is no holding costs in the EFC and the brick-and-mortar stores, since we can modify the feasible set of prices \( \Omega_z \) accordingly to account for holding costs. Note that the objective function is nonconvex in the decision variables due to the bilinear revenue function.

Constraint (4.1b) ensures week \( t \) sales at a brick-and-mortar store to be less than the week \( t \) store-level brick-and-mortar demand. Constraint (4.1c) ensures week \( t \) zone-level sales through the e-commerce channel is less than the week \( t \) zone-level e-commerce demand. Recall that zone-level channel demands are functions of the e-commerce price and the brick-and-mortar zone price. These constraints are non-linear because of the discrete choice nature of these demand functions.

Constraint (4.1d) ensures that the total brick-and-mortar store sales in zone \( z \) cannot exceed the initial store inventory \( x_{bz} \), where \( Y_{bz} \) is a slack variable denoting the leftover inventory in the store in zone \( z \). Constraint (4.1e) ensures that the total e-commerce sales cannot exceed the initial EFC inventory \( x_e \), where \( Y_e \) is the slack variable denoting the leftover inventory in the EFC.
Constraint (4.1f) sets the e-commerce zone-level sales to be equal to the zone-level sales that have been fulfilled through the EFC plus the zone-level sales that have been fulfilled through brick-and-mortar stores.

Constraint (4.1g) specifies that the channel-zone price trajectories must belong to a feasible set of prices trajectories \( \Omega \). The feasible price \( \Omega \) set may include linear constraints that couple prices across time and across zones or its discrete nature. For example, the retailer may want prices to be strictly decreasing during the clearance period. They may want the e-commerce price to be lowest offered price in each period. Upper and lower bounds are often imposed as a percentage of historically offered prices or as a percentage of competitor prices to ensure the competitiveness of the retailer. The discreteness of \( \Omega \) is due to a finite number of feasible prices, since based on customer psychology, magic number endings (e.g., those ending with $0.99) are important to a retailer and are typically encoded in their business rules. When the retailer re-optimizes prices, these rules also avoid taking very small unnecessary price changes.

Constraint (4.1h) are nonnegativity constraints for the sales variables. Constraint (4.1i) are nonnegativity constraints for the slack variables.

4.2. Salient features of the omni-channel optimization model

In the introduction to this paper, we used a real-world example to demonstrate negative consequences of using existing legacy single-channel pricing models for omnichannel retail. The OCP-X model improves upon single-channel pricing models because of the following salient features:

1. **Introducing store inventory partitions which can be used to determine the quantity of store inventory to reserve for e-commerce fulfillment.** From the optimal solution to the model, store \( l \) can keep a total of \( \sum_{t \in T} \sum_{z \in Z} S^t_{e,l,z} \) of its store inventory in reserve for future fulfillment of e-commerce demand. Hence, the store inventory partitions identify the true amount of store inventory that is “at-risk” for being unsold and hence result in more appropriate brick-and-mortar store pricing.

2. **Identifying a “virtual” e-commerce inventory pool based on the store inventory partitions.** From the optimal solution, the quantity \( x_e + \sum_{t \in T} \sum_{z \in Z} \sum_{l \in \Lambda} S^t_{e,l,z} \) can be interpreted as the total inventory available for fulfilling e-commerce purchases. Note that part of this inventory is distributed over the retailer’s network of brick-and-mortar stores.

3. **Modeling customers’ channel-switching behavior due to channel prices.** In our experiments with the retailer’s data in Section 6, we observe significant channel-switching behavior by customers. Hence, our omni-channel optimization model can significantly increase the retailer’s profit by setting channel prices that reduce channel cannibalization. Moreover, the optimization model uses pricing to control channel demands across multiple zones and cross-channel fulfillment to correct for inventory imbalances in the network of brick-and-mortar stores.
The OCP-X model partitions store inventory, as a result selling each store inventory unit at the maximum revenue, whether through the e-commerce channel or through the brick-and-mortar channel. Hence, the retailer reduces the opportunity cost of selling through a less profitable channel. These reserves can be directly or indirectly maintained by using the notion of value of inventory similar to the on-the-fly fulfillment scheme proposed by Acimovic and Graves (2014). The use of alternative fulfillment schemes can influence the retailers realized profits, which would depend on the price difference between e-commerce and stores, inventory scarcity, and sell-through rates.

4.3. Modeling Extensions
The OCP-X model can also admit various extensions, which we discuss below:

**SFS capacity constraints:** Due to limited resources in-store to handle ship-from-store (SFS) orders, oftentimes, there are capacity restrictions on the SFS fulfillment variables \( S_{e,(z'z)} \) which limit the SFS inventory flows during each time period. These constraints are important during peak (holiday) periods when the retailer’s sales are several times higher than off-peak sales. Another example is when a store is not included in the SFS fulfillment network in which case its capacity constraint is 0.

**Dimensionality reduction:** In the OCP-X model presented, the SFS flows can be interpreted as “virtual” inventory for the central warehouse. Mathematically, since demand is deterministic, an optimal solution is to have SFS flow variables for only the first period. Hence, we can reduce the number of dimensionality of the OCP-X model by only modeling first period SFS flow variables. This dimensionality reduction technique is valid in the absence of tight SFS capacity constraints.

**Buy Online Pickup In Store (BOPS):** Suppose \( \eta_{ez} \) fraction of the e-commerce demand \( D_{ez}^t(p_{bz}^t, p_e^t) \) choose to pickup their order at a brick-and-mortar store in zone \( z \). We can introduce a sales variable \( S_{ez,pl}^t \) for BOPS, and modify constraint (4.1d) such that the total sales (including BOPS) fulfilled from a store is less than store inventory. This sales variable contributes per-unit price \( P_e^t \) to the objective, but at no shipping cost to the retailer. The existing e-commerce sales variables \( S_{ez}^t \) in the model is now interpreted as the e-commerce sales shipped to the customer and constraint (4.1c) is modified with the appropriate demand \( (1 - \eta_{ez}) D_{ez}^t(p_{bz}, p_e^t) \).

**Warehouse to store allocation:** Oftentimes stores receive inventory replenishments mid-season. These inventory shipments impact optimal prices, since prices depend on inventory availability. Therefore, one can introduce inventory shipments as inventory flow variables \( \chi_{el} \) between the warehouse and the stores at a certain transportation cost \( C_{el} \) and jointly optimize for prices as well as inventory variables.

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\(^2\) Sales data from 2012 of one retailer reveals that sales during holiday season is more than ten times the sales during off-peak months.
4.4. MIP reformulation of the omni-channel optimization problem

Due to a non-convex objective function and non-convex constraints, OCP-X is a non-convex optimization problem. As discussed in Section 2, it is has been shown that attraction demand models are known to provide unimodal and convex structure to some special cases of the pricing problem using a market share transformations. Using these transformations, the OCP-X single location problem, a special case, turns out to be convex. We exploit this in the next section to derive closed form solutions and insights. However, the multi-location single period OCP-X model even without inventory constraints or fulfillment considerations is nonlinear and non-convex problem without a unimodal structure (see example in Harsha et al. (2015) who study this special case). Nevertheless, for the case of general attraction demand models (3.4)–(3.5), the OCP-X problem can be reformulated into an equivalent linear mixed integer program (MIP) that can be solved tractably using commercial off-the-shelf solvers such as IBM ILOG CPLEX. The reformulation is along the lines of Harsha et al. (2015) but generalizes it to a multi-period fulfillment setting where additional transformations are required. The transformation introduces lost-share variables, uses the fractional programming transformations proposed by Charnes and Cooper (1962) to overcome the nonlinearity arising from the ratio terms, and then uses the reformulation and linearization technique (RLT) proposed by Sherali and Adams (1999) to eliminate the nonlinearities due to product terms. The RLT transformations exploit the discrete nature of the binary variables, allowing us to recover an exact reformulation of the problem. We describe the transformations and present the reformulation in Appendix A. The run time tractability results of this formulation on real data is provided in Section 6.3.

5. Benefits of pricing and cross-channel fulfillment controls: Insights from special cases

In this section, we provide insights to the general omni-channel optimization model through an analysis of the one and two-zone case. We consider a product sold through two channels by a retailer with a capacitated e-commerce warehouse and one to two brick-and-mortar stores. In this section, we gain the following insights:

- With unlimited capacity, the retailer can earn a higher revenue when customers choose a channel based on prices versus when customers are loyal to one channel. In the former case, the retailer can use price control to increase its total profit and its total market share.

- With a capacitated e-commerce warehouse, as the customer utility from purchasing through the e-commerce channel increases, a retailer relying solely on price controls will set increasingly divergent channel prices, while losing an increasing amount of profit potential due to lost sales. But if the retailer can use pricing and cross-channel fulfillment controls, then it can reduce the
difference in prices across channels and stores, while a large proportion of the lost profit potential can be regained. These insights highlight the benefits from joint price and cross-channel fulfillment controls for omni-channel revenue management with capacity constraints.

Consider the special case of the single zone and single period. Let $p_e$ be the price charged at the e-commerce store, and $p_b$ be the price charged at the brick-and-mortar store. Let $c_e$ be the per-unit cost of e-commerce sales (product cost plus e-commerce fulfillment cost), and let $c_b$ be the per-unit cost of brick-and-mortar sales.\(^3\) Let $N$ denote the total market size of potential customers. Prior to the retailer’s omni-channel strategy, customers would consider only the e-commerce channel or the brick-and-mortar channel, never both. That is, their choice set is $V = \{e\}$ or $V = \{b\}$ along with the no-purchase option. However, after the retailer initiates its omni-channel strategy, customers now consider both channels (i.e., $V = \{e, b\}$ along with no-purchase) before making a purchase.

The purchase probabilities using MNL demand models can be defined using the set $V$ in the two settings as follows, similar to Eq. (3.2) and (3.3):

$$\lambda_i(p_e, p_b) = \frac{e^{\alpha_i - \beta p_i}}{1 + \sum_{i' \in V} e^{\alpha_{i'} - \beta p_{i'}}}, \quad \forall i \in V,$$

$$\lambda_i(p_e, p_b) = 0, \quad \forall i \notin V,$$

$$\lambda_0(p_e, p_b) = \frac{1}{1 + \sum_{i' \in V} e^{\alpha_{i'} - \beta p_{i'}}},$$

where $\lambda_i$ denotes the market share of channel $i$ and $\lambda_0$ the lost share. For simplicity of deriving closed form solutions, we restrict to the case when the price effect $\beta$ is the same in both channels.

**Lemma 1.** When customers do not compare between channels, let $(p_0^e, p_0^b)$ be the optimal channel prices, $\lambda^0$ be the optimal total market share, and $R^0$ be the optimal profit. When customers compare between channels before making a purchase, let $(\tilde{p}_e, \tilde{p}_b)$ be the optimal channel prices, $\tilde{\lambda}$ be the optimal total market share, and $\tilde{R}$ be the optimal profit. Then, under unlimited capacity constraints, we have that $\max \{p_0^b - c_b, p_0^e - c_e\} \leq \tilde{p}_b - c_b = \tilde{p}_e - c_e$ and $\lambda^0 \leq \tilde{\lambda}$. Moreover, $R^0 \leq \tilde{R}$.

All proofs in this section are in Appendix B. Lemma 1 provides an important motivation for retailers turning to omni-channel retailing: an ability to capture a larger total market share by providing more purchasing options to customers, and as a result increasing the total profit. Lemma 1 offers theoretical evidence to the value of omni-channel pricing even when there are no capacity constraints, which is similar to the setting studied by Harsha et al. (2015).

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\(^3\) The introduction of product cost in this setting slightly generalizes the setting considered in the rest of the paper but it allows the reader to compare the results presented here with the non-perishable item literature where discrete-choice models are often used.
Due to the popularity of online shopping, many brick-and-mortar retailers started offering an e-commerce channel. However due to the significant investment cost of e-commerce capacity, many only have a limited capacity dedicated for fulfilling e-commerce sales. Unlike pure e-commerce retailers such as Amazon.com that has more than 100 e-fulfillment centers (EFCs) dedicated for fulfilling e-commerce purchases, many omni-channel retailers only have a few EFCs. We next show that for retailers with limited e-commerce capacity, the increase in the optimal total market share through omni-channel retail is capped under price-only control. However, this is not the case if the retailer can use both price controls and cross-channel fulfillment controls.

Let $\alpha C$ be the maximum capacity of the EFC dedicated for fulfilling e-commerce sales. Let $(1 - \alpha)C$ be the maximum capacity of the brick-and-mortar store for fulfilling brick-and-mortar sales. Further suppose that the total capacity $C$ is greater than the total market size $N$, so that if the retailer was able to perfectly distribute its capacity at no cost, then the optimal channel prices and optimal market share is equal to the uncapacitated case. We assume that for a marginal cross-channel fulfillment cost, $\nu > 0$, the retailer is able to use a unit of its brick-and-mortar capacity to fulfill an e-commerce sale. That is, the retailer can use both price controls and cross-channel fulfillment controls. When $\alpha$ is sufficiently small (i.e., the EFC has limited capacity), then Proposition 1 below gives the optimal channel prices, the optimal channel shares, the optimal cross-channel fulfillment quantity, and the optimal profit as a function of the intrinsic utility from e-commerce purchase, $a_e$.

**Proposition 1.** Suppose there exist channel capacity constraints, and that the retailer has both pricing controls and cross-channel fulfillment controls. Let \((p_e^{(P,F)}, p_b^{(P,F)})\) be the optimal channel prices, \((\lambda_e^{(P,F)}, \lambda_b^{(P,F)})\) be the optimal channel shares, $z^{(P,F)}$ be the optimal cross-channel fulfillment quantity, and $R^{(P,F)}$ be the optimal profit. If $\nu > 0$, $N \leq C$ and $\alpha < N/C$, then

$$
p_e^{(P,F)} - c_e = \begin{cases} \frac{1}{\beta} (1 + \omega(a_e)) , & \forall a_e \leq \bar{a}_e , \\
\frac{1}{\beta} \left( a_e - \beta c_e - \ln \left( \frac{\alpha C}{N - \alpha C} + \frac{\alpha C}{N} W(\sigma) \right) \right) , & \forall a_e \in (\bar{a}_e, \bar{a}_e + \beta \nu) , \\
v + \frac{1}{\beta} \left( 1 + \omega(a_e - \beta \nu) \right) , & \forall a_e > \bar{a}_e + \beta \nu
\end{cases}
$$

$$
p_b^{(P,F)} - c_b = \begin{cases} \frac{1}{\beta} (1 + \omega(a_e)) , & \forall a_e \leq \bar{a}_e , \\
\frac{1}{\beta} \left( \frac{N}{N - \alpha C} + W(\sigma) \right) , & \forall a_e \in (\bar{a}_e, \bar{a}_e + \beta \nu) , \\
\frac{1}{\beta} \left( 1 + \omega(a_e - \beta \nu) \right) , & \forall a_e > \bar{a}_e + \beta \nu
\end{cases}
$$

$$
z^{(P,F)} = \begin{cases} 0 , & \forall a_e \leq \bar{a}_e + \beta \nu , \\
N \pi_e (a_e - \beta \nu) \left( 1 - \frac{1}{1 + \omega(a_e)} \right) - \alpha C , & \forall a_e > \bar{a}_e + \beta \nu ,
\end{cases}
$$

$$
\lambda_e^{(P,F)} = \begin{cases} \pi_e (a_e) \left( 1 - \frac{1}{1 + \omega(a_e)} \right) , & \forall a_e \leq \bar{a}_e , \\
\frac{\alpha C}{N} , & \forall a_e \in (\bar{a}_e, \bar{a}_e + \beta \nu) , \\
\pi_e (a_e - \beta \nu) \left( 1 - \frac{1}{1 + \omega(a_e - \beta \nu)} \right) , & \forall a_e \geq \bar{a}_e + \beta \nu
\end{cases}
$$
where

\[
\sigma := \frac{N}{N - \alpha C} e^{a_b - \beta c_b - \frac{N}{N - a_c}},
\]

\[
\bar{a}_e := a_b - \beta (c_b - c_e) + \ln \left( \frac{\alpha C}{N - \alpha C} \right) + \ln \left( 1 + \frac{N}{(N - \alpha C)W(\sigma)} \right),
\]

\[
\pi_e(a_e) := \frac{e^{a_e - \beta c_e - 1}}{e^{a_e - \beta c_e - 1} + e^{a_b - \beta c_b - 1}},
\]

\[
\omega(a_e) := W(e^{a_e - \beta c_e - 1} + e^{a_b - \beta c_b - 1}).
\]

As can be inferred, when \( v \) approaches infinity, then it is as if the retailer has price-only controls since it would only fulfill e-commerce demand using the EFC capacity. The results in the above proposition and the corollary below are shown to also hold for \( v \to \infty \).

**Corollary 1.** Suppose there exist channel capacity constraints, and that the retailer has pricing and cross-channel fulfillment controls. Let \( N \leq C \) and \( \alpha < N/C \). Then the absolute difference between the retailer’s optimal channel margins is 0 for \( a_e \in (-\infty, \bar{a}_e) \), is linearly increasing in \( a_e \) for \( a_e \in (\bar{a}_e, a_e + \beta v) \), and is equal to \( v \) for \( a_e \in (a_e + \beta v, \infty) \). Additionally, the retailer’s optimal market share is strictly increasing in \( a_e \) for \( a_e \in (-\infty, a_e) \), is equal to a constant for \( a_e \in (\bar{a}_e, a_e + \beta v) \), and is strictly increasing in \( a_e \) for \( a_e \in (a_e + \beta v, \infty) \). Moreover, the difference \( \tilde{R} - R^{(P,F)} \) is increasing in \( a_e \), where \( \lim_{a_e \to -\infty} \left( \tilde{R} - R^{(P,F)} \right) \leq (N - \alpha C)v \). In particular, when the retailer has only pricing control (i.e., the case when \( v \to \infty \)), \( \lim_{a_e \to -\infty} \left( \tilde{R} - R^{(P)} \right) = \infty \), where \( R^{(P)} \) denotes the optimal profit under this control.

We can understand Corollary 1 using the specific example of Fig. 5 which compares a retailer using pure pricing controls (NoCross) versus a retailer using joint pricing and cross-channel fulfillment controls (Cross) as the intrinsic e-commerce utility \( a_e \) increases. In this example, we use \( N = 200, C = 300, \alpha = 0.1, c_e = c_b = 0, v = 1, \beta = 0.8 \) and \( a_b = 2 \). Clearly, the system-wide inventory constraint is never binding since it exceeds the maximum market size, however the e-commerce inventory constraint could potentially be binding. The top-left panel of Fig. 5 shows the optimal channel prices. The bottom-left panel shows the optimal market shares of each channel if the retailer can only use pricing controls to maximize its revenue. The bottom-right panel shows the
optimal market shares if both pricing and cross-channel fulfillment controls are available. Note that e-commerce sales can either be fulfilled from the e-fulfillment center (EFC) or using the more expensive cross-channel option (SFS). The top-right panel shows the “regret”, which we define as the unconstrained optimal profit $\tilde{R}$ minus the constrained optimal profit, normalized by the market size $N$.

From the figure, note that if the e-commerce channel is insufficiently attractive ($a_e \leq 1.31$), then the optimal e-commerce market share is small and the e-commerce capacity constraint is non-binding for NoCross and Cross (see bottom panels, where EFC capacity is 15% of the total market share). Thus, the optimal channel prices are equal to the unconstrained problem with both channel prices being equal (see top-left panel). Note that while $a_e < 1.31$ the retailer can increase its total market share by marginally increasing $a_e$, for example, by promoting the e-commerce store, by making the online store easier to navigate, or by decreasing time to fulfill online purchases. For mid-range values of $a_e$ (1.31 to 2.11), the NoCross and Cross optimal channel prices begin diverging from the unconstrained price (see top-left panel). Note that since the difference between the e-commerce price and the brick price is less than the cross-fulfillment cost $v$ when $a_e \in (1.31, 2.11)$, the Cross e-commerce sales is fulfilled only using e-fulfillment center inventory (see bottom-right panel). When the e-commerce channel is significantly attractive ($a_e \geq 2.11$), a retailer with only
Figure 6 Optimal price control and cross-channel fulfillment control for different regions of the intrinsic e-commerce utility $a_e$ and the cross fulfillment marginal cost $v$.

Pricing control (NoCross) needs to maintain a relatively large price difference between its channels to entice more customers to purchase from the brick channel instead of the capacity-constrained e-commerce channel (see top-left panel). Moreover, note that with only pricing controls, the maximum total market share is flat (see bottom-left panel) even if the retailer continues to make e-commerce more attractive to customers. However, if the retailer can additionally use cross-channel fulfillment controls (Cross), then it can keep channel prices closer together when $a_e \geq 2.11$ (see top-left panel). Since for $a_e \geq 2.11$, the Cross e-commerce price is $v$ units higher than the Cross brick-and-mortar price, then after the EFC inventory has been depleted, the retailer will choose to fulfill e-commerce sales using ship-from-store (SFS). Hence, from the bottom-right panel we observe that the total e-commerce market share can be increased through cross-channel fulfillment (SFS), thus increasing the total market share.

Fig. 5 also shows the normalized regret in the top-right panel. We observe that if the retailer can only use price controls, then the regret can be arbitrarily large (Corollary 1). On the other hand, if the retailer additionally has cross-channel fulfillment controls, the normalized regret is bounded by $(N - \alpha C)v/N$. An implication of this is that, if the retailer had the option to increase its e-commerce dedicated capacity with a fixed cost $K$, where $K > (N - \alpha C)v$, then it is optimal to not choose this option and instead to use the brick-and-mortar channel to fulfill the e-commerce sales unmet by the existing e-commerce capacity.

Based on the closed-form expressions in Proposition 1, we can define regions of $(v, a_e)$ where the retailer should use a price only control and where it should use a joint price and cross-channel fulfillment control (see Fig. 6). The figure shows three distinct regions. If customers receive a low intrinsic utility from the e-commerce channel, then the e-commerce sales will be less than the e-commerce capacity, and the retailer will only use price controls. For larger values of the intrinsic
e-commerce utility, the retailer will only use cross-channel fulfillment if the marginal cost $v$ is sufficiently low. Otherwise, the retailer will only use pure pricing control.

Based on Fig. 5, in the case of a single zone, cross-channel fulfillment allows the retailer to set optimal channel prices closer together thus reducing channel cannibalization. We next discuss that even the price variation across zones decreases for the two-zone case, along with all the insights observed earlier, in Fig. 7. In this example, the two zones are identical except in their inventory levels with $N = 200$, $c_e = c_b = 0$, $a_e = 4$, $a_b = 2$ and $\beta = 0.8$ in each zone. As can be noted, the e-commerce channel is more attractive in non-price attributes. If a unit is shipped within a zone then a $1$ fulfillment cost is incurred, otherwise the fulfillment cost is $1.25$. The available inventory in the e-fulfillment center is $30$, the available inventory in the brick store of Zone A is $135(1 - \gamma)$, and the available inventory in the brick store of Zone B is $135(1 + \gamma)$ for some $\gamma \in [0, 1]$. The parameter $\gamma$ is the inventory imbalance factor, which is $0$ if the brick inventory is equally balanced between zones. Fig. 7 shows the optimal channel prices, normalized regret, NoCross market share, and Cross market share as the inventory imbalance factor increases.

If inventory is unlimited, then it is optimal to set roughly $4$ prices on all channels and zones (top-left panel). Since the e-commerce channel is more attractive, then in this unconstrained case, the e-commerce market share is significantly higher than the brick market share. Now assume the inventory constraints on the channels. Note that the e-commerce capacity constraint is tight (equal to $7.5\%$ of the total market size). If the retailer can only use price controls, then it will set the e-commerce price to be significantly higher than the brick prices to entice customers to purchase from the less-constrained brick channel (top-left panel). The NoCross e-commerce market share is at capacity (see bottom-left panel). The total market share equal to about $50\%$ for $\gamma \leq 0.4$, and is linearly decreasing for $\gamma \geq 0.4$ when the brick inventory constraint in Zone A is tight (Zone A brick inventory is linearly decreasing in $\gamma$). Note that for $\gamma \geq 0.4$, the brick prices in Zone A and Zone B also start diverging due to the tight inventory constraint in Zone A.

On the other hand, if the retailer can employ both pricing and cross-fulfillment controls, then there is a smaller difference in the optimal channel prices as well as small price variation across two zones (see top-left panel of Fig. 7). If $\gamma \leq 0.3$, then the difference between the e-commerce price and brick prices is $1$, the same-zone fulfillment cost. Note that when $\gamma \leq 0.3$, the retailer uses only the same-zone cross-channel fulfillment (see bottom-right panel). Hence, similar to the insight from Fig. 6 for the single-zone case, the difference in the channel prices is equal to the cross-fulfillment cost when this option is being used. For $\gamma \in (0.3, 0.8)$, note that brick inventory

4 Unlike the single-zone special case, there is no closed-form solution to the two-zone omni-channel pricing problem due to nonconvexity as a result of the equal price constraint in e-commerce prices across zones. Hence, we numerically solve for the optimal controls.
in Zone B (Zone A) is linearly increasing (decreasing). Thus cross-fulfillment from Zone A to Zone A is decreasing and is being replaced by cross-fulfillment from Zone B to A. In this region of $\gamma$, also note the slight increase in the e-commerce price and the Zone A brick prices to justify the use of the more expensive cross-zone fulfillment. For $\gamma \geq 0.8$, the Zone A brick inventory constraint becomes tight, thus we observe that the Zone A brick price becomes significantly higher. This is a pure price-control strategy without any cross-channel fulfillment from the brick store in Zone A. The online price in Zone A marginally drops resulting in enticing Zone A customers to purchase online which can be fulfilled from brick store in Zone B. Observe that the overall price variation across the brick locations is smaller in the Cross scenario over the NoCross scenario. This result is quite notable as one can view cross-channel fulfillment control using SFS as a means of stabilizing the prices systemwide, i.e., across channels and even locations.

The Cross total market share is close to 65% for all values of $\gamma$, which is significantly higher than the NoCross market share. Finally note that when comparing the regret between a price-only
control against price and cross-fulfillment controls (top-right panel of Fig. 7), we observe that by using cross-fulfillment controls the retailer can reduce its regret by between 30% to almost 50%.

This section reveals that joint pricing and cross-channel fulfillment controls can significantly increase profits of an omni-channel retailer when it has limited capacity dedicated to e-commerce particularly by reducing the price variation systemwide. Beyond the single-zone special case, however, finding a closed form expression for the optimal channel prices and cross-channel fulfillment controls is difficult due to nonconvexity. In the next section, we demonstrate that the omni-channel pricing and fulfillment planning optimization model introduced in Section 4 results in a significant profit increase in a real-world retail setting through computational experiments on data from a large U.S. retailer.

6. Computational experiments with data from a major omni-channel retailer

In this section, we demonstrate the financial and operational benefits of the omni-channel pricing and fulfillment planning (OCP-X) model for clearance pricing of a major U.S. retailer. The focus of our computational experiments are items that are marked for clearance because (1) inventories are not replenished during the clearance period, and (2) the retailer actively uses price control through markdowns to clear inventory. The current legacy clearance pricing system solves a single-channel optimization model. Therefore, the retailer has to resort to several manual adjustments that were subjective and labor intensive because the post-processing requirements differed by SKU. In these experiments we find an average of 7% increase in sales, a 6-12% increase in clearance revenue as well as faster delivery times for three product categories through OCP-X.

We engaged with the retailer for 6 months. In the first few weeks, we worked with all stakeholders to define the business problem, collect and process the data and selected categories to be analyzed. Thereafter, we designed, implemented and performed the demand estimation and the OCP-X optimization, and finally, developed the business value assessment. The details of the different steps are provided below.

6.1. Data summary

The omni-channel retailer operates more than 1000 stores and one EFC. They provided us with transactional-log (TLOG) and inventory data between January 1, 2014 to December 31, 2014 for three product categories: Notebooks, Tablets, and Tablet Accessories. During this one-year period, there were several SKUs across the three categories that were sold through both channels. However, there were only 195 of these SKUs whose complete clearance period was during January 1, 2014 to December 31, 2014. We isolated the sales data, including the regular and clearance periods, for these 195 SKUs (see Table 2 for the details about the SKUs in the experiment). The clearance
Table 2 Data summary of SKUs used in the computational experiments.

<table>
<thead>
<tr>
<th>Categories</th>
<th># SKUs</th>
<th>% Sales From E-commerce</th>
<th>% E-commerce Ship From Store</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notebooks</td>
<td>100</td>
<td>13.4%</td>
<td>94.6%</td>
</tr>
<tr>
<td>Tablets</td>
<td>45</td>
<td>14.4%</td>
<td>93.7%</td>
</tr>
<tr>
<td>Tablet Accessories</td>
<td>50</td>
<td>5.4%</td>
<td>90.4%</td>
</tr>
<tr>
<td>Total</td>
<td>195</td>
<td>11.4%</td>
<td>93.8%</td>
</tr>
</tbody>
</table>

period typically lasts for 10-12 weeks. On average, about 89% of lifecycle sales made in these categories is through brick-and-mortar store purchase, and about 11% is through the e-commerce channel. Finally, due to limited EFC inventory, a large proportion (about 94%) of the e-commerce sales are fulfilled using inventory shipped from a store. During clearance alone, the e-commerce volume share steeply increases to about 24% and the SFS fraction are close to 100%, highlighting the importance of OCP-X during this period.

We were additionally provided with price information for 18 online competitors. This competitor price data overlapped with 59 SKUs in our study and among those that had an overlap, on average, there were 6 competitors per SKU.

We were also provided with item cost, salvage and several business rules by SKU. The business rules typically included a minimum time between markdowns, minimum and maximum markdown price percentage and price bounds. The retailer’s current fulfillment strategy was to fulfill the item from the EFC, if it had any inventory, and if not, the nearest in-stock store was chosen. Also, no online order was denied until all stores in the network were sold out.

6.2. Demand model calibration

We geo-spatially clustered the retailer’s brick-and-mortar stores into 50 zones using a k-means (k=50) algorithm that takes as input the store locations. Fig. 2 shows the 50 zones we use in the experiments for all the categories. We geo-tag all the transactions in the TLOG data using zones based on the origin of the demand and the fulfillment location. We ignored the buy-online-pickup-instore option since we observed few such transactions in the data for the items analyzed. Using the zone-tagged weekly historical sales by channel obtained along with the other causal features, we estimate the parameters of the omni-channel demand model at the SKU-zone level. This approach accounts for heterogeneity in channel preferences across the geography as depicted in Fig. 2 which corresponds to the 45 SKUs in Tablets category.

The SKUs in our experiments exhibit a distinct product lifecycle (PLC) that represented the baseline popularity of a product over its selling season that begins at time $t_{\text{start}}$ and has a pre-planned exit date of $t_{\text{end}}$. We use functions motivated from the beta distribution to estimate the non-linear PLC curve. Model selection and cross-validation on a variety of training instances yielded
the following market-size model that predicts the customer arrival rate for any week \( t \) over the selling season:

\[
\log(\text{Market Size}_t) = \gamma_0 + \gamma_1 \log(1 + t - t_{\text{start}}) + \gamma_2 \log(1 + t_{\text{end}} - t) + \sum_k \gamma_{3,k} \text{HOLIDAY-VARIABLES}_{k,t},
\]

and the following market-share model to predict the channel shares in week \( t \):

\[
\log(\text{Channel Attraction}_t) = \beta_0 + \beta_1 \text{PRICE}_t + \sum_k \beta_{2,k} \text{PROMOTION-VARIABLES}_{k,t} + \sum_j \beta_{3,j} \text{COMPETITOR-PRICE-VARIABLES} (\text{optional})_{j,t}.
\]

Holiday spikes, if any, are addressed using holiday indicator variables. It was also beneficial to add channel-specific temporal lag effects prior to the holiday weeks in order to model the spike in online gift orders placed earlier due to the lead time of delivery. Promotional indicators, which include whether the product was advertised that week, were also useful. Finally, the competitor prices are introduced as channel-specific attributes, whenever they are available.

A key prediction component for clearance optimization is the end-of-life sales decay. This decay can occur due to one or more factors, including the waning popularity of the product towards the end of life that reduces the customer arrival rate (captured via the \( \gamma_2 \) coefficient of the PLC effect). This decay can increase by inventory depletion effects such as broken assortments or store-presentations, which tends to increase the proportion of no-buys. Furthermore, ship-from-store to fulfill online orders accelerate store inventory depletion in a store. To gauge the incremental impact of low inventory levels on store-sales during the markdown period, we experimented with several threshold-based inventory-effect models (Smith and Achabal 1998, Caro and Gallien 2012). However, we did not observe any significant improvement in prediction quality after incorporating such inventory effects, and a PLC-based market-size prediction model was adequate for our application. A review of the in-store display and sales transaction procedure employed by the retailer indicated that the categories we analyzed were unlikely to be influenced by the broken assortments and the store-presentation effects. Nevertheless, incorporating inventory effects in an omni-channel environment can be useful for relevant product categories such as fashion apparel.

Estimating the decay coefficient, \( \gamma_2 \), for the PLC curve without the full sales history of an item is challenging (e.g., future end-of-life sales curve can be convex, concave or affine). To overcome this problem, we employed the following two-phase procedure to estimate the parameters of the demand model. In the first phase, the average end-of-life sales decay coefficient \( \gamma_2 \) for a representative SKU from the category was estimated using a learning procedure, and employed as a ‘prior’ desired value in the second phase of estimation that is done at a SKU-zone level for all SKUs. Such priors can
also be estimated using historical values of like-SKUs in the same category. The training sales data was primarily used to estimate same-channel and cross-channel price elasticities, holiday effects, as well as the sales-increase phase $\gamma_1$ of the PLC. As we move closer toward the end of the season, and more end-of-life sales data becomes available, the prediction model is recalibrated on a weekly basis using the most recent data, updating all coefficients including the decay coefficient $\gamma_2$ with its previous estimate used as a prior, to produce improved sales forecasts for the remaining weeks.

The market-size, lost-share, and MNL channel shares are estimated using the method described in Subramanian and Harsha (2014). Other alternatives for calibrating MNL models in the presence of censored lost sales data that are compatible with the data scenario and forecasting requirements described in our work can be employed to calibrate the proposed demand model such as the Expectation-Maximization approach (Talluri and Van Ryzin 2004), and two-step method (Newman et al. 2014).

We now discuss the numerical results for our achieved model fit and prediction quality. The prediction results presented here represent a 12 week ahead forecast for the entire clearance period as opposed to iteratively generating rolling weekly sales predictions. This is because the OCP-X model at the beginning of the clearance periods requires an estimate of demand for all periods till the planned end date of an item. Clearly, as we move forward to the end of horizon, the demand predictions for the remaining weeks will need to be revised each period. The forecast quality was measured in terms of the volume weighted mean absolute percentage error (WMAPE). We
observed this to be largely dependent on the sales rates and hence, the level of disaggregation at which the model calibration was performed, which is consistent with the observations in (Caro and Gallien 2012). The achieved out-of-sample WMAPE at the category-chain level was about 22%. This WMAPE value is in close proximity to that observed by Caro and Gallien (2012) who report a 23.8% WMAPE at the category-chain level. At the lowest level of aggregation (SKU-zone level), the sales rate during the clearance period was more than 10 times lower than the mid-season sales rate, resulting in the absolute deviation of the predicted sales rate being 10 ± 5 for brick and 2 ± 1.2 for e-commerce. Fig. 8 is a sample graphical plot of the model fit and predictions (final 12 weeks of sales).

To measure the impact of cross-channel causals, we ran the estimation algorithm by channel to independently predict brick-store and digital channel sales. We observed an improvement of 1.5 percentage points for the brick channel, and 5 percentage points for the digital channel at the SKU-zone level was obtained over a single-channel forecast approach using the same model features and data. In general, across multiple retailers, we observed that the online sales prediction improved significantly after incorporating cross-channel price and promotion effects. Overall, the prediction quality and demand modeling framework was consistent with the goals set by our customer, and was embedded within our proposed optimization framework to calculate optimal prices and inventory partitions.

**Same-channel and cross-channel price elasticities:** We present the average price same-channel and cross-channel elasticity values evaluated at the average channel price for the Tablets category in Table 3. These relatively high elasticity values are typical of markdown settings. Observe also that the cross-channel elasticities are not insignificant. Note also that the cross-elasticities are asymmetric in that the impact of brick prices on the online sales is different from (and tends to be higher than) the impact of the online prices on brick sales. It is indicative of the heterogeneity of the customers shopping in the different channels as well as the volume share of these channels (the absolute change in volume of brick sales is much higher than that for the online channel).

### Table 3: Average same-channel and cross-channel price elasticities for Tablets Category.

<table>
<thead>
<tr>
<th>Channel Sales</th>
<th>Elasticity to brick-and-mortar price</th>
<th>Elasticity to e-commerce price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brick-and-mortar sales</td>
<td>-1.3</td>
<td>0.7</td>
</tr>
<tr>
<td>E-commerce sales</td>
<td>2.8</td>
<td>-3.9</td>
</tr>
</tbody>
</table>

6.3. Optimizing channel prices and inventory partitions with OCP-X

After estimating the omni-channel zone-level demand for each of the 195 clearance SKUs in our experiments, we next solve the omni-channel optimization model OCP-X introduced in Section 4.
A penalty based on distances was used as proxy for shipping cost in order to maintain the priority ordering specified by the retailer. The MIP based OCP-X model along with the business rules were developed as a JAVA API and were evaluated on a MAC computer having 16GB RAM, and an Intel Core i7 processor. CPLEX 12.6.2 with its out-of-box parameter settings was used to solve the resultant MIPs to optimality. The optimization model was solved at the beginning of the clearance period and used up to 20 discretizations in the price space based on the initial price. For each SKU, the optimization model determines the optimal e-commerce price trajectories and the optimal brick-and-mortar zone price trajectories. The optimization model also determines the optimal store inventory partitions reserved for e-commerce fulfillment.

Fig. 9 displays the distribution of the computational run times of the MIPs, the percentage gap of the MIP optimal solution to the LP root node, and the total number of branch-and-bound nodes in solving the MIP problem. We observe that 80% of the instances solved in about 40 seconds, with a gap of at most 1% with the root note LP with little or no branching required.

We next discuss in Fig. 10 the solutions of the OCP-X model by analyzing the optimal markdown prices for the Tablet computer SKU discussed in Fig. 1 in Section 1. Using the single-channel legacy pricing process, the retailer sold 1,552 units of this SKU during a 12-week clearance period, and made more than $202,800 in revenue. By setting channel prices based on the omni-channel optimization model, the calibrated omni-channel demand model predicts that the retailer can increase sales by 11%, corresponding to an increase in markdown revenues during clearance by almost 6%. We discuss next the features of the optimized solution that enable this revenue gain.

The left panels of Fig. 10 plot the actual zone-weighted average channel prices (empty markers) against the OCP-X optimal average channel prices (filled markers) over the 12-week period. Because the OCP-X model reserves a portion of the store inventory for e-commerce fulfillment (dashed
line in top-right panel), the model can effectively reduce the average brick markdown compared to the actual brick prices. Based on our demand parameter estimates for this SKU, we find that the e-commerce demand is highly sensitive to both the e-commerce price and the brick-and-mortar price; on the other hand, the brick-and-mortar demand is relatively inelastic to e-commerce prices. The OCP-X model exploits this asymmetry, and tends to set a lower e-commerce price.

Since the legacy process prescribes steep markdowns at the brick stores (bottom-right panel) from the regular retail price of $146, the actual brick sales is relatively high. However, there is unsold brick inventory remaining by the end of the clearance period (top-right panel), which is due to inelastic store demand in some zones. This unsold inventory cannot be depleted by cross-channel fulfillment during Weeks 45–51, since there is no e-commerce sales due to the high actual e-commerce price. Unlike brick-and-mortar sales, e-commerce sales can be fulfilled from any brick store. In contrast, the OCP-X model sees higher e-commerce sales due to steep e-commerce markdowns. The OCP-X model uses ship-from-store to deplete inventory from locations associated with price-inelastic store demand. As a result, the OCP-X model decreases unsold inventory by 180 units (top-right panel).

We now summarize the results of the experiments on the 195 SKUs under clearance pricing in the period of January 1, 2014 to December 31, 2014. The following table shows the aggregate product category level effect of the OCP-X model on the average channel prices (normalized by regular price), the category-wide sales and the clearance revenue projections:

<table>
<thead>
<tr>
<th>Category</th>
<th>Brick Price</th>
<th>eComm Price</th>
<th>% Change in Sales</th>
<th>% Change in Clearance Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Actual</td>
<td>Opt</td>
<td>eComm  Brick Total</td>
<td></td>
</tr>
<tr>
<td>Notebooks</td>
<td>81% 93%</td>
<td>90% 85%</td>
<td>+16% -1%  +3%</td>
<td>+10%</td>
</tr>
<tr>
<td>Tablets</td>
<td>81% 90%</td>
<td>79% 77%</td>
<td>+15%  +10%  +11%</td>
<td>+12%</td>
</tr>
<tr>
<td>Accessories</td>
<td>81% 93%</td>
<td>90% 85%</td>
<td>+11%  +5%  +6%</td>
<td>+6%</td>
</tr>
</tbody>
</table>
Based on these findings, we make the following observations: (1) the retailer is setting steeper markdowns than optimal in its brick-and-mortar stores, (2) the e-commerce clearance price should be reduced to significantly increase e-commerce sales activity which can be used to deplete unsold store inventory, and (3) the optimal omni-channel prices result in 3% to 11% higher sales (7% on average). The combined effect of the new omni-channel prices and the fulfillment inventories yields an increase in the clearance revenue ranging between 6% and 12%. The dollar figures are not disclosed for confidentiality reasons.

The omni-channel pricing system also predicts more than 21% reduction in lost brick-and-mortar sales opportunities across all categories. This is due to two reasons: better management of demand using omni-channel prices, and directing e-commerce sales away stores that are likely to sell-out in the future. Also there is more than 23% reduction in the unsold clearance inventory across all categories. This is due to e-commerce sales being fulfilled using stagnant brick-and-mortar store inventories.

Recall that we used ship penalties based on distances as a proxy for shipping cost in the OCP-X model. The average penalty per unit SFS is a direct measure of shipping times. Although minimizing this was not our primary goal, we observe a 2.3% reduction in shipping times on average.

Drawing lessons from this computational experiment, it is possible to infer that there is a potential for significant financial benefits by implementing an omni-channel pricing system for clearance pricing. The results were presented to the retailer, and were reviewed by a team consisting of pricing analysts and senior executives for revenue management, all of whom were well-versed with the legacy markdown pricing solution. Their feedback was overwhelmingly positive along with an immediate request for deploying a commercial version of OCP-X. They also pointed that the aside from the financial benefits, OCP-X would substantially reduce labor cost achieved by the automated generation of store-inventory partitions.

A proprietary version of the proposed solution described in the paper was therefore approved for commercial deployment at IBM. We are currently working with IBM Commerce on a pilot project to test the performance of OCP-X in actual implementation and it is producing equally promising results.

7. Conclusion

Many retailers are choosing to integrate their on-line and off-line channels to remain competitive as e-commerce sales is expected to grow in the next few years. For many omni-channel retailers, investing in additional warehouses dedicated for e-commerce fulfillment is prohibitively expensive, hence they rely on a network of brick-and-mortar stores for e-commerce order fulfillment. Moreover, with cross-channel fulfillment, lost e-commerce sales can be avoided even if the e-commerce warehouse is out of stock. However, even as these retailers have increasingly integrated channels, they
still use legacy processes developed for single-channel pricing and inventory planning, potentially resulting in inconsistent prices, channel cannibalization, and lost revenue. In this paper, we have introduced an optimization model which jointly optimizes cross-channel fulfillment inventories and lifecycle channel prices. The key benefits of the model are in directing e-commerce sales away stores that are likely to sell-out towards stores that have slow moving inventories, and better managing channel demands using omni-channel prices.

Revenue management literature has long demonstrated the benefit of pricing controls in maximizing revenue from selling limited inventory in a single channel. Using a special case of the omni-channel model, we demonstrated an important operational benefit of cross-channel fulfillment for a retailer with a tight capacity for its e-commerce warehouse. That is, if the retailer only relies on pricing controls to manage channel demand, then these channel prices can diverge significantly to entice customers to purchase from the uncapacitated brick-and-mortar channel, resulting in channel cannibalization. However, if the retailer can use both cross-channel fulfillment and pricing controls, then it can regain a significant proportion of lost sales from the capacity constraint and significantly reduce differences in channel prices. We closely collaborated with a major U.S. omni-channel retailer in testing our proposed optimization model for clearance pricing of several electronics products. Computational experiments show that during a 10-12 week clearance period, the omni-channel model achieves an average of 7% increase in sales, a 6-12% increase in markdown revenue, and faster delivery times, compared to the actual prices and fulfillment determined by legacy practices.

Appendix A: MIP reformulation of the omni-channel optimization problem

Let the feasible discrete channel prices be denoted by:

\[ \Omega_t^e = \{p_{t,i}^e\} \forall t \in T, \]

\[ \Omega_{t, bz}^b = \{p_{t,i}^{bz}\} \forall z, \forall t \in T. \]

Then the set \( \Omega \) in (4.1g) is the intersection of the linear pricing rules and cartesian product of all sets \( \{\Omega_t^e\}_{t \in T} \) and sets \( \{\Omega_{t, bz}^b\}_{t \in T, z \in Z} \).

To reformulate the problem into a mixed integer linear program, we first introduce the sets of binary decision variables \( \{W_t^e\}_{t \in T, i \in I_t^e} \) and \( \{W_{t, bz}^b\}_{t \in T, z \in Z, i \in I_{t, bz}} \) to the omni-channel pricing problem (4.1). Let \( W_t^e \) be equal to 1 if and only if the e-commerce price at week \( t \) is equal to \( p_{t,i}^e \). Let \( W_{t, bz}^b \) be equal to 1 if and only if the week \( t \) price for brick-and-mortar zone \( z \) is equal to \( p_{t,i}^{bz} \). We also define the following constants:

\[ g_{t}^{e} = f_{t}^{e}(p_{t}^{e}), \forall t \in T, z \in Z, i \in I_{t}^{e}, \]

\[ g_{t}^{bz} = f_{t}^{bz}(p_{t}^{bz}), \forall t \in T, z \in Z, i \in I_{t}^{bz}, \]
where \( f_{bx}(\cdot) \), \( f_{cz}(\cdot) \) are the attraction functions in (3.4)–(3.5). To linearize the non-linear terms in the optimization problem, we additionally introduce the following decision variables:

\[
R_z^t = \frac{1}{1 + \sum_{i \in I^T} g_{bz}^t W_{bz}^t + \sum_{i \in I^T} g_{cz}^t W_{cz}^t}, \quad \forall t \in T, \forall z \in Z, \quad (A.1a)
\]

\[
U_{bzi}^t = R_z^t W_{bzi}^t, \quad \forall t \in T, \forall z \in Z, \forall i \in I^t_{bz}, \quad (A.1b)
\]

\[
U_{ezi}^t = R_z^t W_{ezi}^t, \quad \forall t \in T, \forall z \in Z, \forall i \in I^t_{e}, \quad (A.1c)
\]

\[
V_{bzi}^t = S_{bz}^t W_{bzi}^t, \quad \forall t \in T, \forall z \in Z, \forall i \in I^t_{bz}, \quad (A.1d)
\]

\[
V_{ezi}^t = S_{e}^t W_{ezi}^t, \quad \forall t \in T, \forall z \in Z, \forall i \in I^t_{e}. \quad (A.1e)
\]

The non-convex OCP-X model (4.1) can be reformulated as the linear MIP optimization model (A.2) in Fig. 11 denoted as OCPX-MIP. Constraints (A.2n)–(A.2o) enforce that exactly one price can be chosen for each week, channel, and zone. For any feasible solution, the objective function (A.2a) is equal to the retailer’s omni-channel revenue minus fulfillment costs. Note that constraint (A.2g) is introduced to enforce the identity in (A.1a). Constraints (A.2h) are to enforce the relationship (A.1b), since if \( W_{bzi}^t = 1 \), then the constraints imply \( U_{bzi}^t = R_z^t \) (due to \( R_z^t \leq 1 \)); otherwise if \( W_{bzi}^t = 0 \), then \( U_{bzi}^t = 0 \). Similarly, constraints (A.2i) are equivalent to the relationship (A.1c). Constraints (A.2j) are introduced to enforce the relationship (A.1d), since if \( W_{bzi}^t = 1 \), then the constraints imply \( V_{bzi}^t = S_{bz}^t \) (due to \( S_{bz}^t \leq n_{i}^t g_{bz}^t U_{bz}^t = n_{i}^t g_{bz}^t R_z^t \)); otherwise, if \( W_{bzi}^t = 0 \), then \( V_{bzi}^t = 0 \). Similarly, constraints (A.2k) are equivalent to the relationship (A.1e). Constraints (A.2l) are the RLT constraints that strengthen constraints (A.2h–A.2i) respectively. Similarly, constraints (A.2m) are the RLT constraints that strengthen constraints (A.1d–A.1e). Note that when \( W_{bzi}^t = 1 \) and \( W_{ezi}^t = 1 \), then the right-hand side of constraints (A.2b) and (A.2c) is equal to \( D_{bz}^t(p_{bz}^t, p_{ezi}^t) \) and \( D_{ezi}^t(p_{ezi}^t, p_{ezi}^t) \) respectively. Hence, the general optimization model of Section 4 can be reformulated as a mixed integer program.

**Appendix B: Proofs**

**B.1. Proof of Lemma 1**

Let us first consider the case when customers do not compare between channels and only visit one channel. Let \( \phi \in [0,1] \) be the fraction of customers that browse the online store, and \( 1 - \phi \) the fraction of customers who visit the brick-and-mortar store. For a visitor to the e-commerce store, the purchase probability is

\[
\lambda_e(p_e) = \frac{e^{p_e - p_e}}{1 + e^{p_e - p_e}}.
\]

For a visitor to the brick-and-mortar store, the purchase probability is

\[
\lambda_b(p_b) = \frac{e^{p_b - p_b}}{1 + e^{p_b - p_b}}.
\]

Then, the retailer’s problem of determining channel prices to maximize revenue is equal to

\[
\max_{p_e, p_b} N \phi(p_e - c_e) \lambda_e(p_e) + N (1 - \phi)(p_b - c_b) \lambda_b(p_b)
\]

Note that the problem is non-convex, but we can solve an equivalent convex problem in the purchase probability space \((\lambda_e, \lambda_b)\) by writing the prices as the inverse of the channel purchase probabilities, i.e. \(p_e(\lambda_e), p_b(\lambda_b)\).
Figure 11  MIP reformulation of OCP-X model.

OCP-X-MIP: maximize
\[
\sum_{i \in I} \sum_{t \in T, z \in Z} \left( \sum_{b \in b} p_{b}^{t} V_{b}^{t} + \sum_{i \in I} p_{i}^{t} V_{i}^{t} \right) + q \left( Y_{c} + \sum_{i \in Z} Y_{b} \right) - \sum_{t \in T} \sum_{z, z'} \left( c_{(z, z')} + \sum_{i \in I} c_{(z', z)} S_{i}^{t} \right) \], \quad (A.2a)
\]
subject to
\[
S_{b}^{t} \leq n_{z}^{t} \sum_{i \in I} g_{b}^{t} U_{b}^{t}, \quad \forall z \in Z, \forall t \in T, \quad (A.2b)
\]
\[
S_{b}^{t} \leq n_{z}^{t} \sum_{i \in I} g_{b}^{t} U_{b}^{t}, \quad \forall z \in Z, \forall t \in T, \quad (A.2c)
\]
\[
\sum_{t \in T} \left( S_{b}^{t} + \sum_{z' \in Z} c_{(z, z')} \right) + Y_{b} = x_{b}, \quad \forall z \in Z, \quad (A.2d)
\]
\[
\sum_{t \in T} \sum_{z \in Z} S_{b}^{t} + Y_{c} = x_{c}, \quad (A.2e)
\]
\[
S_{b}^{t} = S_{b}^{t} + \sum_{z' \in Z} S_{b}^{t} \left( z', z \right), \quad \forall z \in Z, \forall t \in T, \quad (A.2f)
\]
\[
R_{z}^{t} + \sum_{i \in I} g_{b}^{t} U_{b}^{t} + \sum_{i \in I} g_{b}^{t} U_{b}^{t} = 1, \quad \forall t \in T, \forall z \in Z, \quad (A.2g)
\]
\[
U_{b}^{t} \leq R_{z}^{t}, \quad U_{b}^{t} \leq W_{b}^{t}, \quad \forall t \in T, \forall z \in Z, \forall i \in I_{b}^{t}, \quad (A.2h)
\]
\[
U_{b}^{t} \leq R_{z}^{t}, \quad U_{b}^{t} \leq W_{b}^{t}, \quad \forall t \in T, \forall z \in Z, \forall i \in I_{b}^{t}, \quad (A.2i)
\]
\[
U_{i}^{t} \leq V_{b}^{t}, \quad V_{b}^{t} \leq W_{b}^{t}, \quad \forall t \in T, \forall z \in Z, \forall i \in I_{b}^{t}, \quad (A.2j)
\]
\[
V_{b}^{t} \leq S_{b}^{t}, \quad V_{b}^{t} \leq S_{b}^{t}, \quad \forall t \in T, \forall z \in Z, \forall i \in I_{b}^{t}, \quad (A.2k)
\]
\[
\sum_{i \in I} U_{i}^{t} = R_{z}^{t}, \quad \sum_{t \in T} U_{b}^{t} = R_{z}^{t}, \quad \forall t \in T, \forall z \in Z \quad (A.2l)
\]
\[
\sum_{i \in I} V_{i}^{t} = S_{b}^{t}, \quad \sum_{t \in T} V_{b}^{t} = S_{b}^{t}, \quad \forall t \in T, \forall z \in Z \quad (A.2m)
\]
\[
\sum_{i \in I} W_{i}^{t} = 1, \quad \forall t \in T, \forall z \in Z \quad (A.2n)
\]
\[
\sum_{i \in I} W_{i}^{t} = 1, \quad \forall t \in T, \quad (A.2o)
\]
\[
W_{b}^{t} \in \{0, 1\}, \quad W_{b}^{t} \in \{0, 1\}, \quad \forall t \in T, \forall z \in Z, \forall i \in I_{b}^{t}, \forall e_{i} \in I_{b}^{t}, \quad (A.2p)
\]
\[
S_{b}^{t}, S_{b}^{t}, S_{b}^{t}, S_{b}^{t}, S_{b}^{t}, S_{b}^{t} \geq 0, \quad \forall z, z' \in Z, t \in T, \quad (A.2q)
\]
\[
Y_{c}, Y_{b} \geq 0, \quad \forall z \in Z. \quad (A.2r)
\]
Note that since the Lambert function shares, we can show that the optimal channel margins are of the purchase probability functions. By taking the first order conditions and solving for the optimal market which is non-convex, can be convexified in the \( W \) where
\[
W(\cdot) = e^{ax - \beta} \left( 1 + W(e^{ax - \beta}) \right),
\]
where \( W(\cdot) \) is the Lambert W function, which is the inverse function of \( f(W) = We^W \). The total optimal market share is
\[
\lambda^0 = 1 - \phi \frac{1 - \phi}{1 + W(e^{ax - \beta c_e})} - \frac{1 - \phi}{1 + W(e^{ax - \beta c_b})}.
\]
Now let us consider the case when all customers compare between the two channels before making a purchase. The probability of purchase from the e-commerce and brick-and-mortar channels are
\[
\lambda_e(p_e, p_b) = \frac{e^{ax - \beta p_e}}{1 + e^{ax - \beta p_e} + e^{ax - \beta p_b}},
\]
\[
\lambda_b(p_e, p_b) = \frac{e^{ax - \beta p_b}}{1 + e^{ax - \beta p_e} + e^{ax - \beta p_b}},
\]
and the corresponding inverse is
\[
p_e(\lambda_e, \lambda_b) = \frac{1}{\beta} \left( a_e - \log \left( \frac{\lambda_e}{1 - \lambda_e - \lambda_b} \right) \right), \quad (B.1)
\]
\[
p_b(\lambda_e, \lambda_b) = \frac{1}{\beta} \left( a_b - \log \left( \frac{\lambda_b}{1 - \lambda_e - \lambda_b} \right) \right). \quad (B.2)
\]
Then the retailer’s pricing problem is
\[
\max_{p_e, p_b} N(p_e - c_e)\lambda_e(p_e, p_b) + N(p_b - c_b)\lambda_b(p_e, p_b),
\]
which is non-convex, can be convexified in the \((\lambda_e, \lambda_b)\) variables by writing the channel prices as the inverse of the purchase probability functions. By taking the first order conditions and solving for the optimal market shares, we can show that the optimal channel margins are
\[
\tilde{p}_e - c_e = \tilde{p}_b - c_b = \frac{1}{\beta} \left( 1 + W(e^{ax - \beta c_e} + e^{ax - \beta c_b}) \right).
\]
That is, if customers compare prices between channels, it is optimal for the retailer to price such that the channel margins are equal. This is a known result for pricing under MNL demand (Aydin and Porteus 2008).

The optimal channel market shares are equal to
\[
\tilde{\lambda}_e = \frac{e^{ax - \beta c_e}}{e^{ax - \beta c_e} + e^{ax - \beta c_b}} \left( 1 - \frac{1}{1 + W(e^{ax - \beta c_e} + e^{ax - \beta c_b})} \right), \quad (B.3a)
\]
\[
\tilde{\lambda}_b = \frac{e^{ax - \beta c_b}}{e^{ax - \beta c_e} + e^{ax - \beta c_b}} \left( 1 - \frac{1}{1 + W(e^{ax - \beta c_e} + e^{ax - \beta c_b})} \right). \quad (B.3b)
\]
Hence, the optimal market share is equal to
\[
\tilde{\lambda} = 1 - \frac{1}{1 + W(e^{ax - \beta c_e} + e^{ax - \beta c_b})}.
\]
Note that since the Lambert function \( W(\cdot) \) is an increasing function, then it follows that
\[
\max \{W(e^{ax - \beta c_e}), W(e^{ax - \beta c_b})\} \leq W(e^{ax - \beta c_e} + e^{ax - \beta c_b}).
\]
Hence, \( \max \{p^0_e - c_e, p^0_b - c_b\} = \tilde{p}_e - c_e = \tilde{p}_b - c_b, \) proving the first part of the lemma. Also, for all \( \phi \in [0, 1] \)
\[
\lambda^0(\phi) \leq \max \left\{ 1 - \frac{1}{1 + W(e^{ax - \beta c_e})}, 1 - \frac{1}{1 + W(e^{ax - \beta c_b})} \right\} \leq \tilde{\lambda}.
\]
Since both the optimal channel margins and optimal total market share are greater if customers compare between channels, then it follows that \( R^0 \leq \tilde{R}. \) Q.E.D.
B.2. Proof of Proposition 1

We will prove the proposition for the case of a finite \( v \), when the retailer has the option to use both pricing and cross-fulfillment controls. Note that for the case of \( v \to \infty \), the problem is equivalent to the retailer only having pricing controls, and the proof for the optimal channel prices follows a similar logic as the finite case.

The retailer’s problem is equivalent to

\[
\max_{p_e, p_b, z} N(p_e - c_e) \lambda_e(p_e, p_b) + N(p_b - c_b) \lambda_b(p_e, p_b) - v z,
\]

subject to

\[
N \lambda_e(p_e, p_b) \leq \alpha C + z,
\]

\[
N \lambda_b(p_e, p_b) \leq (1 - \alpha) C - z,
\]

\[
0 \leq z \leq (1 - \alpha) C.
\]

The above problem is non-convex. However, we can write it into an equivalent convex problem in the \((\lambda_e, \lambda_b)\) space by letting the channel prices equal the inverse of the channel share functions (B.2). The equivalent convex formulation is

\[
\max_{\lambda_e, \lambda_b, z} \frac{N \lambda_e}{\beta} \left( a_e - \beta c_e - \ln \left( \frac{\lambda_e}{1 - \lambda_e - \lambda_b} \right) \right) + \frac{N \lambda_b}{\beta} \left( a_b - \beta c_b - \ln \left( \frac{\lambda_b}{1 - \lambda_e - \lambda_b} \right) \right) - v z,
\]

subject to

\[
N \lambda_e \leq \alpha C + z,
\]

\[
N \lambda_b \leq (1 - \alpha) C - z,
\]

\[
0 \leq z \leq (1 - \alpha) C.
\]

The KKT conditions of the retailer’s problem are given by the following:

\[
a_e - \beta c_e - 1 - \frac{\lambda_e}{1 - \lambda_e - \lambda_b} - \frac{\lambda_b}{1 - \lambda_e - \lambda_b} - \ln \left( \frac{\lambda_e}{1 - \lambda_e - \lambda_b} \right) = \beta \mu_e, \quad (B.4a)
\]

\[
a_b - \beta c_b - 1 - \frac{\lambda_e}{1 - \lambda_e - \lambda_b} - \frac{\lambda_b}{1 - \lambda_e - \lambda_b} - \ln \left( \frac{\lambda_b}{1 - \lambda_e - \lambda_b} \right) = \beta \mu_b, \quad (B.4b)
\]

\[
v - \mu_e + \mu_b + \theta_1 - \theta_2 = 0, \quad (B.4c)
\]

\[
\mu_e, \mu_b, \theta_1, \theta_2 \geq 0, \quad (B.4d)
\]

\[
N \lambda_e - z \leq \alpha C, \quad (B.4e)
\]

\[
N \lambda_b + z \leq (1 - \alpha) C, \quad (B.4f)
\]

\[
0 \leq z \leq (1 - \alpha) C, \quad (B.4g)
\]

\[
(N \lambda_e - z - \alpha C) \mu_e = 0, \quad (B.4h)
\]

\[
(N \lambda_b + z - (1 - \alpha) C) \mu_b = 0, \quad (B.4i)
\]

\[
(z - (1 - \alpha) C) \theta_1 = 0, \quad (B.4j)
\]

\[
z \theta_2 = 0, \quad (B.4k)
\]

Since the retailer’s problem is convex, then the KKT conditions are necessary and sufficient for optimality. Hence if we find variables \( \lambda_e^\star, \lambda_b^\star, z^\star, \mu_e^\star, \mu_b^\star, \theta_1^\star, \theta_2^\star \) that satisfy the KKT conditions (B.4), then \( \lambda_e^\star, \lambda_b^\star, z^\star \) is the optimal solution for the retailer’s problem.
Solution form # 1: Consider the following candidate solution:

\[ \lambda_e = \tilde{\lambda}_e, \quad \lambda_b = \tilde{\lambda}_b, \]

\[ z = \mu_e = \mu_b = \theta_1 = 0, \quad \theta_2 = v, \]

where \( \tilde{\lambda}_e, \tilde{\lambda}_b \) are defined in (B.3). Aside from conditions (B.4e)–(B.4f) which require some care to verify, it is trivial to check that the remaining KKT conditions are satisfied by the above solution.

First we show that the candidate solution satisfies (B.4e), or \( \tilde{\lambda}_e \leq aC/N \). In order to show this, we define the function

\[ L(x) := \left( 1 - \frac{e^{a - \beta c - x} - 1}{e^{a - \beta c - x} + e^{x - \beta c - 1}} \right) \left( 1 - \frac{1}{1 + W(e^{a - \beta c - x} + e^{x - \beta c - 1})} \right), \quad x \geq 0. \quad (B.5) \]

Note that \( L \) is an increasing function in \( x \), and \( L(a) = \tilde{\lambda}_e \). Hence, if we can show that \( x = \bar{a}_e \) is the root of the equation \( L(x) = aC/N \) (where \( \bar{a}_e \) is defined in the proposition), then the candidate solution satisfies the KKT conditions for \( a_e \leq \bar{a}_e \), since \( \lambda_e = L(a_e) \leq L(\bar{a}_e) = aC/N \).

First, note that after rearranging terms, \( L(x) = aC/N \) is equivalent to

\[ W(e^{x - \beta c - 1} + e^{a - \beta c - 1}) = aC \frac{e^{x - \beta c - 1} + e^{a - \beta c - 1}}{(N - aC) e^{x - \beta c - 1} - aC e^{a - \beta c - 1}} \quad (B.6) \]

By definition of the Lambert \( W \) function, \( y = W(y) e^y \). Hence, letting \( y = e^{x - \beta c - 1} + e^{a - \beta c - 1}, \) by using the Lambert-\( W \) function identity and applying (B.6), we have that

\[ \frac{e^{x - \beta c - 1} + e^{a - \beta c - 1}}{y} = \frac{aC}{(N - aC)} - \frac{e^{x - \beta c - 1} + e^{a - \beta c - 1}}{e^{x - \beta c - 1} - aC e^{a - \beta c - 1}} \exp \left( \frac{aC}{(N - aC)} - \frac{e^{x - \beta c - 1} + e^{a - \beta c - 1}}{e^{x - \beta c - 1} - aC e^{a - \beta c - 1}} \right). \quad (B.7) \]

Rearranging terms in (B.7), we have

\[ \exp \left( \frac{aC}{(N - aC)} - \frac{N e^{a - \beta c - 1}}{e^{x - \beta c - 1} - aC e^{a - \beta c - 1}} \right) = \left( \frac{N - aC}{aC} e^{x - \beta c - 1} - e^{a - \beta c - 1} \right) e^{- \frac{aC}{N - aC}}. \quad (B.8) \]

Now let us define \( \zeta \) as the term inside the exponential of the left-hand side of the equation (B.8). Multiplying both sides of (B.8) by \( \zeta \), we have that

\[ \zeta e^{\zeta} = \frac{N}{N - aC} e^{a - \beta c - \frac{N}{N - aC}}. \quad (B.9) \]

Again using the Lambert \( W \) function identity, we have that (B.9) is equivalent to

\[ W\left( \frac{N}{N - aC} e^{a - \beta c - \frac{N}{N - aC}} \right) = \frac{aC}{N - aC} - \frac{N e^{a - \beta c - 1}}{e^{x - \beta c - 1} - aC e^{a - \beta c - 1}}; \quad (B.10) \]

where note that the right-hand side of (B.10) is equal to \( \zeta \). Plugging in \( x = \bar{a}_e \) into (B.10), we can verify that \( \bar{a}_e \) solves the equation (B.10), and hence is the root for the equation \( L(x) = aC/N \).

Now it remains to show that the candidate solution satisfies (B.4f), or \( \tilde{\lambda}_b \leq (1 - \alpha)C/N \). We know that \( \hat{\lambda}_b + \tilde{\lambda}_e \leq 1 \) and that \( \tilde{\lambda}_e \leq aC/N \). Therefore, \( \lambda_b \leq 1 - aC/N \leq (1 - \alpha)C/N \) with the last inequality because \( N \leq C \).

Thus, solution form # 1 is optimal for the retailer’s problem if \( a_e \leq \bar{a}_e \).
Solution form # 2: Consider the following candidate solution:

\[ \lambda_e = \alpha C/N, \ z = \mu_b = \theta_1 = 0, \]

\[ \lambda_b = \left(1 - \frac{\alpha C}{N}\right) \left(1 - \frac{N}{N + (N - \alpha C)W(\sigma)}\right), \]

\[ \mu_e = \frac{1}{\beta} \left( a_e - \beta c_e - \frac{N}{N - \alpha C} - W(\sigma) - \ln \left(\frac{\alpha C}{N - \alpha C} + \frac{\alpha C}{N} W(\sigma)\right)\right), \]

\[ \theta_2 = -\frac{1}{\beta} \left( a_e - \beta c_e - \frac{N}{N - \alpha C} - W(\sigma) - \ln \left(\frac{\alpha C}{N - \alpha C} + \frac{\alpha C}{N} W(\sigma)\right)\right)\]

where \( \sigma = \frac{N}{N - \alpha C} \exp \left(a_b - \beta c_b - \frac{N}{N - \alpha C}\right) \). We can check that this solution satisfies (B.4a)–(B.4c).

Next, we need to check the nonnegativity conditions (B.4d). Note that \( \mu_e \geq 0 \) if and only if

\[ e^{a_e - \beta c_e} - \frac{N}{N - \alpha C} e^{-W(\sigma)} \geq \alpha C + \frac{\alpha C}{N - \alpha C} W(\sigma). \]  

(B.12)

Plugging in the Lambert W relationship \( e^{-W(\sigma)} = W(\sigma)/\sigma \) into (B.12), and rearranging the inequality, we have that \( \mu_e \geq 0 \) if and only if

\[ \frac{N - \alpha C}{\alpha C} \left(\frac{(N - \alpha C)e^{a_e - \beta c_e - 1} - \alpha C e^{a_e - \beta c_e - 1}}{N e^{a_e - \beta c_e - 1}}\right) \geq \frac{1}{W\left(\frac{N}{N - \alpha C} \exp \left(a_b - \beta c_b - \frac{N}{N - \alpha C}\right)\right)} \]  

(B.13)

Note that the left-hand side of inequality (B.13) is increasing in \( a_e \). Moreover, for \( a_e = \bar{a}_e \), inequality (B.13) is tight because \( \bar{a}_e \) is the root of (B.10). Hence, we need \( a_e \geq \bar{a}_e \) for the candidate solution to satisfy \( \mu_e \geq 0 \).

Next, we need to check that \( \theta_2 \geq 0 \). Note that this is true if and only if

\[ 0 \geq \frac{1}{\beta} \left( a_e - \beta v - \frac{N}{N - \alpha C} - W(\sigma) - \ln \left(\frac{\alpha C}{N - \alpha C} + \frac{\alpha C}{N} W(\sigma)\right)\right) \]  

(B.14)

Note that the right-hand side of (B.14) is increasing in \( a_e \). Moreover, for \( a_e = \bar{a}_e + \beta v \), the right-hand side of (B.14) is equal to zero. Hence, we need \( a_e \leq \bar{a}_e + \beta v \) for the candidate solution to satisfy \( \theta_2 \geq 0 \).

Hence, KKT condition (B.4d) is satisfied for \( a_e \in (\bar{a}_e, \bar{a}_e + \beta v) \). Condition (B.4e) is clearly satisfied by the candidate solution. Note that since \( N \leq C \) from the condition of the proposition, we have that \( \lambda_b \leq 1 - \alpha C/N \leq (1 - \alpha)C/N \), hence KKT condition (B.4f) holds. Finally, it is trivial to check that the remaining KKT conditions (B.4g)–(B.4k) are satisfied.

Thus, solution form # 2 is optimal for \( a_e \in (\bar{a}_e, \bar{a}_e + \beta v) \).

Solution form # 3: Consider the following candidate solution:

\[ \lambda_e = \frac{e^{a_e - \beta c_e - 1} - \beta v}{e^{a_e - \beta c_e - 1} - \beta v + e^{a_b - \beta c_b - 1}} \left(1 - \frac{1}{1 + W\left(e^{a_e - \beta c_e - 1} - \beta v + e^{a_b - \beta c_b - 1}\right)}\right), \]

\[ \lambda_b = \frac{e^{a_e - \beta c_e - 1} - \beta v + e^{a_b - \beta c_b - 1}}{e^{a_e - \beta c_e - 1} - \beta v + e^{a_b - \beta c_b - 1}} \left(1 - \frac{1}{1 + W\left(e^{a_e - \beta c_e - 1} - \beta v + e^{a_b - \beta c_b - 1}\right)}\right), \]

\[ \mu_e = v, \ \mu_b = \theta_1 = \theta_2 = 0, \]

\[ z = N\lambda_e - \alpha C, \]

We can verify that the candidate solution satisfies KKT conditions (B.4a)–(B.4b). It is also trivial to check that (B.4c)–(B.4e) holds. To verify that condition (B.4f) holds, note that

\[ N\lambda_b + z = N (\lambda_e + \lambda_b) - \alpha C = N \left(1 - \frac{1}{W\left(e^{a_e - \beta c_e - 1} - \beta v + e^{a_b - \beta c_b - 1}\right)}\right) - \alpha C \]

\[ \leq N - \alpha C \leq (1 - \alpha)C. \]
To check that condition (B.4g) holds, note that $z \leq N - \alpha C \leq (1 - \alpha)C$. In order to verify nonnegativity of $z$, note that

$$z = \frac{N e^{\alpha - \beta c_a} - 1 - \beta v}{e^{\alpha - \beta c_a} + e^a - e^{-\beta c_a} - 1} \left( 1 - \frac{1}{1 + W(e^{\alpha - \beta c_a} + e^a - e^{-\beta c_a} - 1)} \right) - \alpha C. \quad (B.15)$$

The right-hand side of (B.15) is equal to zero for $a_e = \bar{a}_e + \beta v$ (since $\bar{a}_e$ is the root of $L(x) = \alpha C/N$). Moreover, the right-hand side is increasing in $a_e$. Hence, condition (B.4g) holds for $a_e \geq \bar{a}_e + \beta v$. The remaining KKT conditions (B.4h)–(B.4k) trivially holds for the candidate solution.

Thus, solution form # 3 is optimal for the retailer’s problem if $a_e \geq \bar{a}_e + \beta v$. Q.E.D.

### B.3. Proof of Corollary 1

Except for the last two statements of the corollary, all the other statements can be proved trivially by using the closed-form expression for the optimal channel margins and the optimal channel shares in Proposition 1. We will next prove the second last statement. From the proof of Lemma 1, we have

$$\tilde{R} = \frac{1}{\beta} N \omega(a_e).$$

Thus, based on Proposition 1 it follows that the difference between the unconstrained optimal profit and the constrained optimal profit is

$$D^{(P,F)}(a_e) := \tilde{R} - R^{(P,F)} = \begin{cases} 0, \\ \frac{1}{\beta} \left( N \omega(a_e) - \alpha C \right), & \forall a_e \leq \bar{a}_e, \\ \frac{1}{\beta} N \omega(a_e) - \frac{1}{\beta} N \omega(a_e - \beta v) - \alpha C v, & \forall a_e > \bar{a}_e + \beta v \end{cases}$$

Note that for $\bar{a}_e \leq a_e \leq \bar{a}_e + \beta v$, we have that

$$\frac{dd^{(P,F)}}{da_e} = \frac{1}{\beta} \left( N \omega'(a_e) - \alpha C \right) = \frac{1}{\beta} \left( N e^{\alpha - \beta c_a} - e^a - \beta v \right) \frac{W(e^{\alpha - \beta c_a} + e^a - e^{-\beta c_a} - 1) - \alpha C}{W(e^{\alpha - \beta c_a} + e^a - e^{-\beta c_a} - 1) + \alpha C} = \frac{1}{\beta} \left( N \lambda_e - \alpha C \right) 
geq 0$$

where the second equality follows from $\omega'(a_e) = e^{\alpha - \beta c_a} W'(e^{\alpha - \beta c_a} + e^a - e^{-\beta c_a} - 1)$ and using the derivative of the Lambert-$W$ function, $W'(x) = \frac{W(x)}{x(1+W(x))}$. The third equality follows from the definition of $\lambda_e$ in (B.3).

The inequality follows from $\lambda_e = L(a_e) = \alpha C/N$ for $a_e > \bar{a}_e$, where $L(\cdot)$ is defined in (B.5).

For the case of $a_e > \bar{a}_e + \beta v$, note that

$$\frac{dd^{(P,F)}}{da_e} = \frac{N}{\beta} \left( \omega'(a_e) - \omega'(a_e - \beta v) \right) = \frac{N}{\beta} \left( \lambda_e(a_e) - \lambda_e(a_e - \beta v) \right) \geq 0$$

where the inequality follows because $\lambda_e(\cdot)$ is an increasing function.

Finally, note that since the Lambert-$W$ function is concave, then $\omega(\cdot)$ is also concave. This implies that for any $a_e$,

$$\omega(a_e) \leq \omega(a_e - \beta v) + \beta v \omega'(a_e - \beta v) = \omega(a_e - \beta v) + \beta v \lambda_e(a_e - \beta v).$$
Hence, for $a_e > \bar{a}_e + \beta v$,

$$D^{(P,F)}(a_e) = \frac{N}{\beta} (\omega(a_e) - \omega(a_e - \beta v)) - v\alpha C$$

$$\leq Nv\lambda_e (a_e - \beta v) - \alpha Cv$$

Hence, taking the limit on both sides, and noting that $\lim_{a_e \to \infty} \lambda_e(a_e) = 1$, we have that

$$\lim_{a_e \to \infty} D^{(P,F)}(a_e) \leq (N - \alpha C) v.$$

In particular, with pure pricing controls, we can also show that $D^{(P)}(a_e)$ is convex increasing in $a_e$ since

$$\frac{d^2 D^{(P)}}{da^2} = \frac{N}{\beta} \lambda'(a_e) \geq 0,$$

since we can check that $\lambda'(a_e) \geq 0$ for all $a_e$. Choose an arbitrary $\hat{a}_e > \bar{a}_e$ be such that $m = \left. \frac{dD^{(P)}}{da_e}\right|_{a_e=\hat{a}_e} > 0$. Define $D(a_e) := m(a_e - \hat{a}_e) + D^{(P)}(\hat{a}_e)$, which is the tangent of $D^{(P)}$ at $a_e = \hat{a}_e$. Note that since $D^{(P)}$ is convex and increasing, we have that it is bounded below by the tangent at $\hat{a}_e$, i.e.,

$$D^{(P)}(a_e) \geq m(a_e - \hat{a}_e) + D^{(P)}(\hat{a}_e).$$

Taking the limit on both sides as $a_e \to \infty$, we have that $\lim_{a_e \to \infty} D^{(P)}(a_e) = \infty$. Q.E.D.

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References


