MARKOV RANDOM FIELDS, STOCHASTIC QUANTIZATION AND IMAGE ANALYSIS[1]

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1 Introduction

Markov random fields based on the lattice Z^2 have been extensively used in image analysis in a Bayesian framework as a-priori models for the intensity field and on the dual lattice $(Z^2)*$ as models for boundaries. The choice of these models has usually been based on algorithmic considerations in order to exploit the local structure inherent in Markov fields. No fundamental justification has been offered for the use of Markov random fields (see, for example, GEMAN-GEMAN [1984], MARROQUIN-MITTER-POGGIO [1987]). It is well known that there is a one-one correspondence between Markov fields and Gibbs fields on a lattice and the Markov Field is simulated by creating a Markov chain whose invariant measure is precisely the Gibbs measure. There are many ways to perform this simulation and one such way is the celebrated Metropolis Algorithm. This is also the basic idea behind Stochastic Quantization. We thus see that if the use of Markov Random fields in the context of Image Analysis can be given some fundamental justification then there is a remarkable connection between Probabilistic Image Analysis, Statistical Mechanics and Lattice-based Euclidean Quantum Field Theory. We may thus expect ideas of Statistical Mechanics and Euclidean Quantum Field Theory to have a bearing on Image Analysis and in the other direction we may hope that problems in image analysis (especially problems of inference on geometrical structures) may have some influence on statistical physics.

This paper deals with the issues just described and above all it suggests a program of research for dealing with the fundamental issues of probabilistic image analysis.

What is the fundamental problem of Image Analysis? It may be stated as follows: Given noisy images (possibly stereo) of the visual world in motion, represent and recognize "structures" in some invariant manner. For example, if the structures of concern are three-dimensional rigid objects then we want this recognition to be invariant under the action of the euclidean group. Thus symmetries play an important role in the whole process.

[1]This research has been supported by the Air Force Office of Scientific Research under grant AFOSR 89-0276 and by the Army Research Office under grant ARO DAAL03-86-K-0171 (Center for Intelligent Control Systems)

R. Spigler (ed.), Applied and Industrial Mathematics, 101–109.

Ultimately, we want to identify the mechanisms (circuits) which perform the full recognition task. There are three aspects to this problem: a) the identification and representation of a priori knowledge of the visual world without which the recognition cannot take place; b) the extraction of knowledge from the imprecise images available about the visual world and c) understanding the correct interaction between a priori knowledge and the imprecise data available in the form of images. It is this formulation of the problematique that suggests a Bayesian framework. In this paper we deal only with some partial aspects of b) and c) and it is here that the connection with statistical physics enters.

The visual world is a continuous world and if we believe that it is important to capture symmetries then mathematical formulations should not be on the lattice $\mathcal{L}$ but should be based on $\mathbf{R}^2$. Even if we are interested in algorithms on a digital machine which necessarily involve discretization, then in order to capture the symmetries in the limit of lattice spacing going to zero attention must be paid to the discrete formulations of the problem (see for example, KULKARNI-MITTER-RICHARSON [1990]).

This paper is organized as follows. In section 2 we discuss Markov fields and Euclidean fields and state the conjecture regarding Osterwalder-Schrader fields. Section 3 is concerned with a variational problem in Image Analysis and its probabilistic interpretation and it shows how these ideas are related to those of Section 2. Finally in Section 4 we discuss Stochastic Quantization.

2 Markov Fields and Euclidean Fields. (NELSON [1973])

Let $\mathbf{R}^d$ denote Euchidean d-dimensional space ($d \geq 2$) and let $\mathcal{S}(\mathbf{R}^d)$ denote Schwartz space. If V is a topological vector space, a linear process over V is a stochastic process φ indexed by V which is linear and such that if $f_\alpha \to f$ in V then $\varphi(f_\alpha) \to \varphi(f)$ in measure. This implies that $\varphi(f)$ for $f \in V$ is a random variable over $(\Omega, \mathcal{F}, \mu)$.

Let φ be a linear process over $\mathcal{S}(\mathbf{R}^d)$. If $\Lambda \subset \mathbf{R}^d$ is open, $\mathcal{A}(\Lambda)$ be the σ-algebra generated by $\varphi(f)$ with support of $f \subseteq \Lambda$ and if Λ is any subset of $\mathbf{R}^d$ let $\mathcal{A}(\Lambda) = \bigcap_{\Lambda' \supset \Lambda} \mathcal{A}(\Lambda')$ where Λ' ranges over all open sets containing Λ. $\mathcal{A}(\Lambda)$ will also denote the set of all random variables which are measurable with respect to $\mathcal{A}(\Lambda)$. Let Λ^c denote the complement of Λ and $\partial\Lambda$ denote the boundary of Λ.

A Markov field on $\mathbf{R}^\mathbf{d}$ is a linear process φ over $\mathcal{S}(\mathbf{R}^d)$ such that whenever Λ is open in $\mathbf{R}^d$ and α is an integrable random variable in $\mathcal{A}(\Lambda)$ then $E(\alpha|\mathcal{A}(\Lambda^c)) = E(\alpha|\mathcal{A}(\partial\Lambda))$. Here E denotes conditional expectation.

Let $O(d)$ be the Euclidean group of $\mathbf{R}^d$. By a representation T of $O(d)$ on some probability space $(\Omega, \mathcal{F}, \mu)$ we mean a homomorphism $\eta \to T(\eta)$ of $O(d)$ into the group of automorphisms of the measure algebra and this group acts in a natural way on random variables.

A Euclidean field is a Markov field φ over $\mathcal{S}(\mathbf{R}^d)$ together with a representation T of $O(d)$ on $(\Omega, \mathcal{F}, \mu)$ such that $\forall f \in \mathcal{S}(\mathbf{R}^d)$ and $\eta \in O(d)$

$$\text{(Covariance)} \quad T(\eta)\varphi(f) = \varphi(f \circ \eta^{-1}), \tag{2.1}$$

and if ρ in the reflection in the hyperplane $\mathbf{R}^{d-1}$

$$T(\rho)\alpha = \alpha,\ \alpha \in \mathcal{A}(\mathbf{R}^{d-l}). \tag{2.2}$$

Consider the mapping

$$\begin{aligned} S_n &: (\mathbf{R}^d)^n \longrightarrow C \\ &: S_n(f_1, \cdots, f_n) = E(\varphi(f_l) \cdots \varphi(f_n)) \end{aligned}$$

and assume it is continuous. By the Schwartz Kernel Theorem, there exists a distribution $\mathcal{S}_n \in \mathcal{S}'(\mathbf{R}^{dn})$ such that $S_n(f_1, \cdots, f_n) = \mathcal{S}_n(f_l \otimes \cdots \otimes f_n)$.

Nelson proves that the sequence of distributions $\mathcal{S}_n$ satisfy euclidean invariance, symmetry and Osterwalder-Schrader positivity, namely, if Λ is the half-space $x^d > 0$, so that $\partial\Lambda$ is the hyperplane $\mathbf{R}^{d-1}$ and Λ^c is the half space $x^d \leq 0$ then $E[(T(p)\bar{\alpha})\alpha] \geq 0$ where $\alpha \in \mathcal{A}(\Lambda)$ and $T(p)\bar{\alpha} \in \mathcal{A}(\Lambda^c)$.

An example of such a euclidean field is obtained as follows. Let $\mathcal{S}_{\mathbf{R}}(\mathbf{R}^d)$ be the real Schwartz space, let $m > 0$ and let H be the real Hilbert space completion of $\mathcal{S}_{\mathbf{R}}(\mathbf{R}^d)$ with respect to the scaler product $< g, (-\Delta + m^2)^{-1} f >$ where Δ is the Laplace operator. Let φ be the unit Gaussian on H, i.e. φ is a real Gaussian process indexed by H with mean zero and covariance given by the scalar product on H. Extend φ to the complexification of H by linearity. Restricted to $\mathcal{S}(\mathbf{R}^d)$, φ is a linear process over it. Nelson proves that φ is a Euclidean field, that is it is Markov and satisfies euclidean invariance, symmetry and Osterwalder-Schrader positivity. Indeed the Markov property, covariance property and reflection implies Osterwalder-Schrader positivity.

Non-gaussian random fields can be constructed using Multiplicative functionals or measure transformations. Let φ be a Markov field over $\mathcal{S}(\mathbf{R}^d)$ with the underlying probability space $(\Omega, \mathcal{F}, \mu)$. We say, that a random variable β is multiplicative if for every open cover $\{\Lambda_i\}$ of $\mathbf{R}^d$, there exists strictly positive β_i in $\mathcal{A}(\Lambda_i)$ with $\beta = \prod_i \beta_i$. We shall see later how we construct $P(\varphi)_2$ fields using these ideas.

To see the connections with Gibbs density let us proceed formally. Let

$$H_0(x) = \frac{1}{2}((\nabla\varphi(x))^2 + m^2(\varphi(x))^2)$$

and consider the formal expression

$$\exp(-\int H_0(x)dx) \prod_{x\in\mathbf{R}^2} d\varphi(x).$$

The rigorous interpretation of this expression is as a Gaussian measure μ_c on $\mathcal{S}'(\mathbf{R}^2)$ with mean zero and covariance $C = (-\Delta + m^2)^{-1}$. This measure corresponds to the free euclidean field. Non-gaussian measures are obtained by considering formal expressions

$$\exp(-\int_{even} [H_0(x) + \lambda P(\varphi(x))]dx) \prod_{x\in\mathbf{R}^2} d\varphi(x)$$

where P is an even polynomial and λ is a coupling constant. The above corresponds to the canonical Gibbs density for transverse vibrations of an elastic membrane subject to the non-linear restoring force $F = -m^2\varphi(x) - \lambda P'(\varphi(x))$, (after integration over momentum variables $\hat{\varphi}(x)$), and $'$ denotes derivative).

Now if φ is the unit Gaussian process over $\mathcal{S}(\mathbf{R}^2)$, then $P(\varphi)$, for an even polynomial does not make sense. To fix ideas let us consider the case where $P(\varphi) = \varphi^4$. It turns out, however that

$$:\varphi^4:(g) \triangleq \int g(x) : \varphi(x) : dx \text{ for } g \in L' \bigcap L^\infty$$

has a well-defined meaning in a limiting sense in $L^2(\Omega, \mathcal{F}, \mu)$, where $:\varphi(x)^4:$ is an element of the 4th Homogeneous class. In a more concrete way, $:\varphi^3 :\triangleq \varphi^3 - 3E(\varphi^2)\Phi$ and this is well-defined. Now if we consider

$$\beta = \frac{\exp(-\int g(x) : \varphi^4(x) : dx)}{E(\exp(-\int g(x) : \varphi^4(x) : dx)}$$

then β is a multiplicative random variable and we are able to construct the non-gaussian measure

$$d\mu = \frac{\exp(-\int :\varphi^4 : dx)}{E\exp(-\int :\varphi^4 : dx)} d\mu_C.$$

This measure is the so-called $(\varphi^4)_2$-measure.

The proof that $(\varphi^4)_2$-measure defines a Markov field is surprisingly difficult and has been accomplished only recently (see ALBEVERIO, HOEGH-KROHN and ZEGARLINSKI [1989]). It depends on the complicated theory of local specifications, related Gibbs states and cluster expansions. On the other hand the property of Osterwalder-Schrader positivity is much easier to verify. With a view to proving the Global Markov property and for problems in Image Analysis we advance the following conjecture.

We define an Osterwalder-Schrader field (O-S) on $\mathbf{R}^2$ to be a linear process φ over $\mathcal{S}(\mathbf{R}^2)$ which satisfies the Euclidean covariance property (2.1) and Osterwalder-Schrader positivity. For example an Osterwalder-Schrader field may be obtained by considering a function of a Markov field. We conjecture that every Osterwalder-Schrader field can be obtained as a function of a Markov Field. We call such a Markov field a Hidden Markov Field for the O-S field.

Let us see this conjecture in the familiar context of stochastic processes. Let $(\varphi_t | t \in \mathbf{R})$, $\varphi : \Omega \to V$ be a stochastic process, which is continuous in probability, stationary, symmetric (i.e. φ_t and φ_{-t} are stochastically equivalent). It is O-S positive if $\forall\ 0 \leq t_1 \leq \cdots \leq t_n$, $\forall$ $f : V^n \to C$, bounded Borel (V is a topological vector space) we have

$$< f(\varphi_{t_1}, \cdots, \varphi_{t_n}), \bar{f}(\varphi_{-t_1}, \cdots, \varphi_{-t_n}) >_{L^2(\Omega, \mathcal{F}, P)} \geq 0.$$

Now, let $\tilde{\varphi}_t : \Omega \to \tilde{V}$, be a stationary, symmetric Markov process and let $\psi : \tilde{V} \to V$ be bounded and Borel.

Define a new process by

$$\varphi_t(\omega) = \psi(\tilde{\varphi}_t(\omega)).$$

$\tilde{\varphi}_t$ is called a Markov extension of φ_t. φ_t is symmetric and stationary and satisfies O-S postivity but is not necessarily Markov. The conjecture would be:

Does every stationary, O-S process have a Markov extension?

This conjecture in discrete-time is not true (cf. ARVESON [1986]) but in the form stated previously may be true.

(It is clear that every bounded Borel function of a symmetric, stationary Markov process is O-S positive).

3 Probabilistic View of Image Segmentation

We think of a noisy image as a function $g : \Omega \to \mathbf{R}$ where $\Omega \subset \mathbf{R}^2$ is a bounded, open set. We assume $g \in L^\infty(\Omega)$. A variational formulation of the image segmentation problem due to Mumford and Shah (cf. MUMFORD-SHAH [1989]) is as follows. Approximate g by a function f and a closed set $\Gamma \subset \bar{\Omega}$ such that the following energy function is minimized:

$$E(f,\Gamma) = \beta \int_\Omega |f-g|^2 dx + \int_{\Omega\backslash\Gamma} |\nabla f|^2 dx + \alpha H^1(\Gamma). \tag{3.1}$$

Here f is required to be in $W^{1,2}(\Omega\backslash\Gamma)$, $H^1(T)$ denotes the 1-dimensional Hausdorff measure of Γ and β, α are positive constants. We would like to give a probabilistic interpretation of $E(f,\Gamma)$ by considering

$$\exp(-\int_{\Omega\backslash\Gamma} |\nabla f|^2 dx - \alpha H^1(\Gamma))$$

as a Gibbs density with respect to a suitable reference measure which is to serve as a prior measure on (f,Γ) and $\exp(-\int |g-f|^2 dx)$ as a likelihood function of g given f. The choice of the energy functional is dictated by the requirement that f should approximate g in the L^2-sense, it should be smooth away from the boundaries Γ and the total length of the boundary should be short. Note that for fixed Γ, $\exp(-\int_{\Omega/\Gamma} |\nabla f|^2 dx) \prod_{x\in\Omega\backslash\Gamma} df(x)$ would have a rigorous interpretation as a Gaussian measure μ_C with mean zero and covariance $C = (-\Delta)^{-1}$ on $\mathcal{S}'(\Omega/\Gamma)$. Using the recent work of Surgailis (cf. ARAK-SURGAILIS [1989]) we can give a rigorous interpretation as a density to the following:

$$\exp(-\int_{\Omega\backslash\bigcup_{i=1}^{n}\Gamma_i} |\Delta f|^2 dx - 2\sum_{i=1}^{n} \ell(\Gamma_i) - \varphi(n))$$

which, for an appropriate $\varphi(n)$, is a density over $f \in W_0^{2,2}(\Omega\backslash\bigcup_{i=1}^{n}\Gamma_i)$, n and a set of straight lines Γ_i which form together with $\partial\Omega$ a polygonal partition of Ω (cf. MITTER-ZEITOUNI [1990]). We outline here the basic result of Surgailis which constructs a measure on closed polygonal partitions of Ω.

Let Ω be a closed convex subset of $\mathbf{R}^2$ with smooth boundary. In $\mathbf{R}^2$, choose coordinates (t,x) such that, for all $y \in \Omega$, $t(y)$, $x(y) > 0$. Let $\mathcal{L}_\Omega$ denote the lines which intersect Ω, each line $\ell \in \mathcal{L}_\Omega$ is parameterized by its distance from the origin ρ_ℓ and the angle it forms with the $t = 0$ axis, α_ℓ.

Let $\mu(d\ell)$ be a uniform measure on the set $\alpha_\ell, \rho_\ell | \ell \in \mathcal{L}_\Omega$. The Poisson point process with intensity $\mu(d\ell)$ will be denoted μ^Ω, and the measure it induces on the boundary $\partial\Omega$ by the hitting points $\{(x_\ell, t_\ell) | t_\ell = \inf\{t | \ell \in \Omega\}\}$ is again a Poisson point process on the triple (x, t, ν) with intensity $\mu^{\partial\Omega}$. Here and in the sequel, ν denotes the velocity of the particle, i.e. the tangent of the angle formed by the trajectory and the t axis.

For any line $\ell \in \mathcal{L}_\Omega$, let ν_ℓ denote the slope of ℓ. Clearly, $\mu(d\ell)$ can be considered as a measure on ν_ℓ and x_ℓ, the intersection of ℓ with the x axis. In the sequel, we consider the measure $\mu(d\nu, dx)$ obtained from the uniform measure $\mu(d\ell)$ on α, ρ, i.e.

$$\mu(d\ell) = \mu(d\nu, dx) = dx \frac{d\nu}{(1+\nu^2)^{\frac{1}{2}}}. \tag{3.2}$$

Inside Ω, construct a point process on the quadruple (t, y, ν', ν'') with intensity

$$\mu^P(dt, dy, d\nu', d\nu'') = |\nu' - \nu''| dy dt \frac{d\nu'}{(1+(\nu')^2)^{\frac{3}{2}}} \frac{d\nu''}{(1+(\nu'')^2)^{\frac{3}{2}}}. \tag{3.3}$$

Finally, construct a random partition of Ω as follows:

Pick up on $\partial\Omega$, n_0 triples (t, x, ν) according to the law $\mu^{\partial\Omega}$, and inside Ω, n_1 quadruples (t, y, ν', ν'') according to the Poisson process with intensity μ^P. At each of those points, start a line of slope ν (two lines of slopes ν', ν'' in the case of interior points) and evolve ν according to the Markov transition law

$$P(\nu_{t+dt} \in du | \nu_t = \nu) = |u - \nu| \frac{du dt}{(1+u^2)^{\frac{3}{2}}}. \tag{3.4}$$

Finally, at each intersection of lines (when viewing it in the direction of growing t) kill the intersected lines. Clearly, such dynamics describe a random partition of Ω by polygons, c.f. fig. 1. The basic result of Arak and Surgailis is:

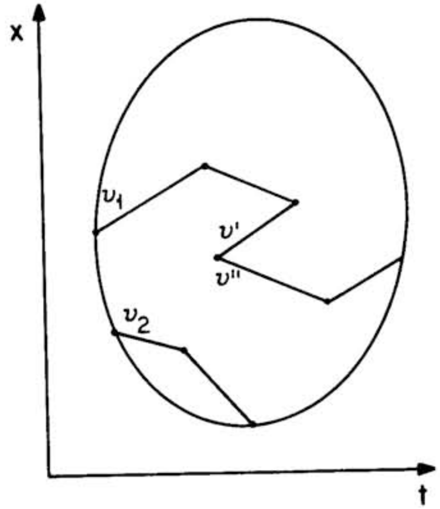

Figure 1: Random Polygonal Parition

Lemma 3.1

$$P(n, \ell \in d\ell_i, i = 1, \cdots, n) = \frac{1}{n!}\mu(d\ell_1)\cdots\mu(d\ell_n)\exp(-2\sum_{i=1}^{n}\mathcal{L}(\ell_i)) \tag{3.5}$$

where $\mathcal{L}(\ell_i)$ denotes the length of the i-th segment.

Note that due to the presence of n in (3.5), one can't consider (3.5) directly as a candidate for a density: indeed, if one were to consider $P(n, \ell, \in \underline{\ell}_i \pm \epsilon)$, the required normalization constant (as $\epsilon \to 0$) would have depended on n and therefore, a path with no jumps will be infinitely more likely than a path with one jump.

One way out of this problem is by using an appropriate definition: Let

$$\begin{aligned} Z^n &\triangleq \int_{\mathcal{L}_D} \cdots \int_{\mathcal{L}_D} P(n, \ell_i \in d\ell_i, i = 1, \cdots, n) \\ Z &\triangleq \sum_{n=0}^{\infty} Z^n. \end{aligned}$$

$(\frac{Z^n}{Z})$ is the probability of having n lines in a specific partition. Now, one may define:

Definition The prior density of a partition (n, ℓ_i) is given by

$$p(n, \ell_i) = (\frac{Z^n}{Z})\lim_{\epsilon\to 0}\frac{P(n, \ell_i \in \ell_i \pm \epsilon, i = 1, \cdots, n)}{2\epsilon^{2n}} \tag{3.6}$$

Combining these ideas with some of the ideas contained in Dembo-Zeitouni (cf. DEMBO-ZEITOUNI [submitted]) the desired result referred to before can be obtained. The more general problem of interpreting $\exp(-\int |\nabla f|^2 dx - H^1(\Gamma))$ as a density remains open.

The interpretation of $\beta\int_\Omega |f - g|^2 dx + \int_{\Omega\setminus\Gamma} |\nabla f|^2 dx + \alpha H^1(\Gamma)$ as a posterior density accomplishes something important. It frees us from obtaining Maximum A Posteriori estimates of (f, Γ) via minimization of the above functional. We can in principle obtain other estimates such as conditional mean estimates. Indeed for closed, convex partitions of Ω it opens up the possibility of doing inference on geometries via Monte-Carlo simulations.

4 Stochastic Quantization (see BORKAR-CHARI-MITTER [1988] for Stochastic Quantization of $(\varphi^4)_2$ fields).

The problem of stochastic quantization for the energy functional (3.1) is to create a Markov process whose invariant measure is $\exp(-E(f, \Gamma))$. In this section we wish to suggest a program for achieving this goal. We concentrate on the prior density $\exp(-\int_\Omega |\nabla f|^2 dx - \alpha H^1(\Gamma))$. The first step in the procedure is to replace the above energy functional appearing in the density by an approximating functional

$$\mathcal{E}_n(f, v) = \int_\Omega |\nabla f|^2(1 - v^2)^n dx + \alpha\int_\Omega((|\nabla f|^2 + |\nabla v|^2)(1 - v^2)^n + \frac{n^2 v^2}{16})dx. \tag{4.1}$$

In (4.1) the variable $v(x) \in [0,1]$ should be thought of as a control variable, which controls the gradient of f and depends on the discontinuity set Γ, n is a parameter which tends to infinity. The above expression makes sense for $f, v \in W^{1,2}(\Omega)$. $1-(1-v^2)^n$ approximates smoothed neighbourhoods of Γ and the approximate boundaries can be identified with $(1-v^2)^n \simeq 0$. If we denote by $\mu_n(B) = 2(n+1)\int_B v(1-v^2)^n|\nabla v|dx$, then essentially $\mu_n(\Omega) \to H^4(\Gamma)$ in a weak sense. For details of this approximation scheme see AMBROSIO-TORTORELLI [1990].

The first step in the program would be to identify $\exp(-\mathcal{E}_n(f,v)) \prod_{x\in\mathbb{R}^2} d(f(x), v(x)) \triangleq d\mu_n$ as a measure in a suitable distribution space. The covariances involved will have appropriate boundary conditions. One can then study the weak convergence of this measure $d\mu_n$ as $n \to \infty$. The functional derivatives of $\mathcal{E}_n$ with respect to f, v can be computed:

$$\frac{\delta\mathcal{E}_n}{\delta f} = -\nabla\cdot(\nabla f\cdot(1-v^2)^n) \quad (4.2)$$

$$\frac{\delta\mathcal{E}_n}{\delta v} = -\alpha\nabla\cdot(\nabla v\cdot(1-v^2)^n) + n(|\nabla f|^2+\alpha|\nabla v|^2)(1-v^2)^n v + \frac{\alpha n^2}{16}v \quad (4.3)$$

(There are additional terms involving the normal derivatives).

In analogy with the study of stochastic quantization of $(\varphi^4)_2$ fields, the problem of stochastic quantization for fixed n is the study of the coupled pair of stochastic differential equations

$$df(t) = -\frac{\delta\mathcal{E}_n}{\delta f}\cdot dt + dw(t) \quad (4.4)$$

$$dv(t) = -\frac{\delta\mathcal{E}_n}{\delta v}\cdot dt + dv(t) \quad (4.5)$$

where $w(\cdot)$ and $v(\cdot)$ are infinite-dimensional independent Brownian motions.

5 Conclusions

The conceptual program outlined in this paper is quite general and may be applied to other variational problems arising in Image Analysis, for example, those involving curvature terms. These variational problems may be important to obtain non-overlapping segmentations of images with a view to identifying occluded regions.

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