Scattering Theory, Unitary Dilations and Gaussian Processes

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I. INTRODUCTION

The themes of modeling and representation of linear deterministic and stochastic systems have dominated much of systems theory in the last thirty years. In the critical period of this development attention was focussed on the reconciliation between the input-output (external) and state space (internal) points of view of systems. A central result in this development is the statement that a minimal (in the sense of dimension) state space realization of linear finite-dimensional system is unique (up to isomorphism) and corresponds to one which is both controllable and observable. In recent years Jan Willems has forcefully argued that this input-output-state space view is narrow and inadequate to deal with models of dynamical systems arising out of physics, econometrics as well as stochastic processes (notably Markov processes). For one thing, there is no natural identification of what is an input and what is an output in these systems. For another, the need to fix the initial state as an equilibrium state in

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conventional realization theory is unnatural and leads to conceptual difficulties. For a detailed exposition of this work see [23].

The other important theme in systems theory is one of optimization and approximation. Models of systems obtained from physical principles or data are often of high dimension. When these models are used for prediction and feedback control it is necessary to obtain approximate models of much lower order so that the algorithms for prediction and/or control are computationally tractable. There is an important question here as to what is the appropriate representation on which the approximation (reduction) process should be carried out. Zames [24] has argued that this approximation should be done on an input-output basis and an internal representation of the approximate input-output map could then be obtained for the purpose of prediction and control. This argument rests on the notion that two systems may be near each other in an input-output sense (for example in $L^1$-topology) and yet may drastically differ dimensionally in their internal representations. The two processes of approximation and internal representation do not, in general, commute and working with the input-output representation for the needs of approximation is a more stable operation.

There is an apparent contradiction here since we have just argued that the input-output view so prevalent in the early days of systems theory is not an appropriate one. Fortunately, Scattering Theory as developed by Adamjan, Arov [1] and Lax and Phillips [15] and the theory of Abstract Hankel Operators comes to the rescue here. The work of Adamjan, Arov and Krein (for example, [2]) and Ball and Helton [6] provide a mathematical framework for dealing with representation and approximation issues in a Hilbert space setting in a rigorous manner. In the systems theory context, Scattering Theory for Gaussian Processes was investigated in the doctoral thesis of Y. Avriel at M.I.T. [cf. 5] and Scattering Theory and Approximation of Linear Systems has been investigated by Willems himself in his own framework for dynamical Systems [22]. A state space view of Hankel approximation has been provided by Glover [13] in an important paper.

Intimately connected with the theory of Scattering is the theory of minimal unitary dilations of contraction semigroups contracting strongly to zero. Indeed, the theorem of Nagy asserts that every contraction semigroup contracting strongly to zero has a (unique) minimal unitary dilation. This theorem has a physical interpretation of coupling a dissipative system to a heat bath so that the resulting composite system is conservative [cf. 17]. The dual of this question, namely, how certain observables of an infinite-dimensional conservative system can exhibit dissipative behavior has been investigated by Picci by using the theory of stochastic realization [cf. 20].

This semi-expository paper consists of two parts. In the first part we describe in an essentially self-contained manner the Scattering Theory associated with stationary Gaussian processes. This is done in discrete time to avoid certain
technical difficulties. The new contribution in this part of this paper is the result which states that for completely non-deterministic stationary Gaussian processes the spectral density can be recovered (up to unitary isomorphism) from the Hankel operator induced by the scattering function associated with the process. This result to some extent justifies using this Hankel operator for model reduction in stochastic systems. We then show the relationship of this scattering view to the theory of Unitary Dilations and Markovian representations. These latter ideas are all contained in the work of Lindquist and Picci [cf.18] and Foias and Fraho [10]. The exposition serves to show that if we restrict ourselves to a $L^2$-theory these representation questions for stochastic (and deterministic) systems are nothing else but a version of Scattering Theory à la Adamjan, Arov and Krein. It is worth mentioning that the Scattering Function plays an important role in the parametrization of the unit ball of the quotient space $L^\infty/\mathbb{H}^\infty$ and the corresponding extension problem for Hankel operators.

The second part of the paper is concerned with the theory of minimal unitary dilations of contraction semi-groups, its relation to positive definite functions on a group and the theory of open systems [8]. We also present a new construction of a unitary dilation of contraction semi-groups which makes evident the coupling to white noise (heat bath) which is implicit in the construction of the dilation.

2.1 NOTATION

$\mathbb{Z}$ stands for the set of integers, $\delta(n)$ for the indicator function of $\{0\} \subseteq \mathbb{Z}$, $\mathbb{C}$ the complex numbers, and for $a \in \mathbb{C}$, $\overline{a}$ denotes the complex conjugate of $a$. For a matrix $A = (a_{ij})_{i,j=1}^p$ we denote by $A^*$ the Hermitian conjugate of $A$: $A^* = (\overline{a_{ji}})_{i,j=1}^p$, $b_{ij} = + \overline{a_{ji}}$, and by $A^t$ its transposition. For a family of subsets $\{M_j\}_{j \in J}$ of Hilbert space $H$, we denote by $\bigvee M_j$ the smallest closed linear manifold (subspace) that includes each $M_j$, and by $A\bigvee M_j$ the largest subspace contained in each of them (their intersection). $\overline{M_j}$ denotes the closure of $M_j$ in $H$. For subspaces $M, N$ of $H$, $M \Theta N$ denotes the orthogonal complement of $N$ in $M$. For a countable family $\{M_j\}_{j \in J}$ of mutually orthogonal subspaces: $M_i \perp M_j \iff i \neq j$, we let $\Sigma_i \Theta M_j$ be their orthogonal sum. $P_M$ stands for the orthogonal projection of $H$ onto the subspace $M$. For a bounded linear operator $A : H_1 \to H_2$ of Hilbert space $H_1$ into $H_2$, we denote by $[A]$ the matrix of $A$ with respect to specified orthonormal bases in $H_1, H_2$.

$A^* : H_1 \to H_2$ denotes the adjoint of $A$. AIM stands for the restriction of $A$ to the subspace $M \subset H_1$. $B(H_1, H_2)$ denotes the Banach space of all bounded linear operators from $H_1$ into $H_2$ with $B(H) = B(H, H)$.

By $\ell_2(\cdot, \cdot, \cdot; \mathbb{N})$ we denote the usual Hilbert space of sequences $\{h_j\}_{j \in \mathbb{N}}$ with values in (the Hilbert space) $\mathbb{N}$ for which $\sum \|h_j\|_N^2 < \infty$. $\ell_2(0, \cdot, \cdot; \mathbb{N})$, $\ell_2(\cdot, \cdot, -1; \mathbb{N})$ are seen naturally as subspaces of $\ell_2(\cdot, \cdot, \cdot; \mathbb{N})$. $L_2, L_\infty$ will denote respectively the Lebesgue spaces on the circle $T = \{e^{i\lambda} : \lambda \in [-\pi, \pi]\}$ (with respect to the normalized Lebesgue measure $\frac{d\lambda}{2\pi}$ of square integrable, essentially bounded complex valued functions). Each function can be viewed as defined on $[-\pi, \pi]$. Similarly for the spaces $L_2(C^p), L_\infty(C^p)$ of functions $f$ taking values in $(C^p)$ for which $\|f(e^{i\lambda})\|_{C^p} \in L_p$, $\|f(e^{i\lambda})\|_{C^p} \in L_\infty$ respectively. $L_\infty(B(C^p))$ is defined analogously for weakly measurable, $B(C^p)$ valued functions $f$ for which $\text{ess. sup} \|f(e^{i\lambda})\|_{B(C^p)} : \lambda \in [-\pi, \pi] < \infty$. $H_2^\perp$ are the subspaces of $L_2$ defined by

$$H_2^\perp = \{f \in L_2 : \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\lambda})e^{-in\lambda} d\lambda = 0, n = -1, -2, \ldots \}$$

and we have the orthogonal decomposition $L_2 = H_2^\perp \oplus H_2^\perp$. Each $f \in H_2^\perp$ having a Fourier series

$$f(e^{i\lambda}) = \sum_{n=0}^{\infty} a_ne^{in\lambda}$$

generates the function

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$
belonging to the Hardy class $H_2$ of functions $g(z)$ holomorphic in $1 < |z| < 1$ and such that

$$\|g\|_{H_2} = \sup_{\theta \in C} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(e^{i\theta})|^2 d\theta \right]^{1/2} < \infty.$$ 

Moreover, the (a.e. existing) radial limit $g(e^{i\lambda})$ of $g(z)$ equals $f(e^{i\lambda})$ a.e. and $\|f\|_{L^2_2} = \|g\|_{H_2}$. The function $g(z)$ is seen as the analytic extension of $f \in H^+_2$ to the unit disc $|z| < 1$ and is denoted by $f(z)$. We identify $H^+_2$ and $H_2$ and denote them commonly by $H_2$. Using the conjugation with respect to the unit circle ($z \mapsto \frac{1}{z}$) by the reflection principle for $f \in H_2 \subseteq L^2$ the function $\overline{f}$ defined by $\overline{f(e^{i\lambda})} = \overline{f(e^{-i\lambda})}$ has an analytic extension to $|z| > 1$ for $f(\frac{1}{z})$ which we again denote by $\overline{f}$. The space $H_2 = \{f \in L^2 : f \circ \pi \in H_2 \}$ is the space of functions $f \in L^2$ having an analytic extension to the exterior of the disc $|z| < 1$ and we have

$$\|f\|_{L^2_2} = \sup_{\theta \in C} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta \right]^{1/2}.$$ 

$f \in H_2$ are called conjugate analytic.

Analogously for the Banach space $L^\infty$, we have the subspaces $H^+_\infty = H^+_2 \subseteq L^\infty$ of functions $f \in L^\infty$ having an analytic extension $f(z)$ to $|z| < 1$ with

$$\|f\|_{L^\infty} = \sup_{|z| < 1} |f(z)| = \|f\|_{H^+_\infty}.$$ 

Similarly, for the Hilbert space $L^2_2(C^p)$, we have the subspaces $H^+_2(C^p) = H_2(C^p)$, $H_2^+(C^p)$ with the orthogonal decomposition $L^2_2(C^p) = H_2(C^p) \oplus H_2^+(C^p)$. In $L^\infty_\infty(B(C^p))$, again $H^+_\infty(B(C^p))$ is defined as the subspace of functions in $L^\infty_\infty(B(C^p))$ whose negatively indexed (matrix valued) Fourier coefficients vanish. For $\theta \in H^+_\infty(B(C^p))$

the function $\theta^*\hat{\theta} = \{\theta(e^{i\lambda})\}_{\lambda \in \mathbb{R}}$ is identified with its analytic extension $\theta^* = \{\theta(e^{i\lambda})\}_{\lambda \in \mathbb{R}}$ to $|z| > 1$.

A function $f \in H_\infty(B(C^p))$ is called inner if $\|f(e^{i\lambda})\|_2 = 1$ a.e. Similarly for $\theta \in H^+_\infty(B(C^p))$ if $\theta(e^{i\lambda})$ is unitary a.e. $f \in H_2$ is called outer if $\overline{\pi(f)} = H_2$ where $\pi$ denotes the function on $T$ defined by $\pi(e^{i\lambda}) = e^{i\lambda}$. For $\phi \in L^\infty_\infty(B(C^p))$ the Toeplitz operator $T_\phi : H_2(C^p) \to H_2(C^p)$ whose matrix is block Toeplitz with respect to the standard basis $\{e^{ik\lambda}e_1, e^{ik\lambda}e_2, \ldots, e^{ik\lambda}e_p\}_{k \geq 0}$, $\{e_1, e_2, \ldots, e_p\}$, the standard basis in $C^p$, is defined by $T_\phi = \pi_\phi(\hat{\phi})$ where $\pi_\phi$ is the Riesz projection of $L^2_2(C^p)$ onto $H_2^+(C^p)$. $H_2$ will denote the Hankel operator (with block Hankel matrix with respect to the standard bases in $H_2(C^p)$, $H_2^+(C^p)$), $H_\phi : H_2(C^p) \to H_2^+(C^p)$ defined by $T_\phi = \pi_\phi(\hat{\phi})$, $\pi_\phi$ being the Riesz projection of $L^2_2(C^p)$ onto $H_2^+(C^p)$. The convention we employ regarding a Hankel operator $\hat{\phi}$ acting from $H^+_2(C^p)$ into $H_2^+(C^p)$ is not in accordance with the one employed in systems theory, where we act on $H^+_2(C^p)$ into $H_2(C^p)$, $\hat{\phi} = \pi_\phi(\hat{\phi})$. It, however, conforms to that employed by Adamjan-Arov-Krein and enables us to use their results without modification, as well as to refer to them.

2.2 SCATTERING THEORY

Let $H$ be a complex separable Hilbert space and let $U$ be a unitary operator on $H$. A subspace $D_+$ is said to be outgoing for $(U, H)$ if it satisfies

(i) $UD_+ \subseteq D_+$

(2.2.1) (ii) $\sum U^n D_+ = \{0\}$
A subspace \( D \) for which

(i) \( \bigoplus_{\infty} U^n D = \mathbb{H} \)

(ii) \( \bigoplus_{\infty} U^n D = \{0\} \)

(iii) \( \bigoplus_{\infty} U^n D = H \)

is said to be incoming for \((H,U)\).

2.2.1 DEFINITION. A quadruple \((U,H,D,D)\) satisfying (2.2.1) is said to be a scattering system.

2.2.2 THEOREM (Translation Representation Theorem [19, Th. II.1.1]). Let \((U,H,D)\) be outgoing. Then there exists a Hilbert space \( N_\perp \) and a unitary map \( r_+ \) of \( H \) onto \( \ell_2(\infty,\infty) \) such that

(i) \( r_+ D = \ell_2(\infty,\infty) \)

(ii) \( U_+ = r_+ U r_+^{-1} \)

is the right shift operator on \( \ell_2(\infty,\infty) \). This representation is unique up to automorphism of \( N_\perp \).

Proof (Standard (cf. [15, p. 77])). We give the proof to establish various quantities introduced later. By (2.2.1), (ii) the operator \( U D \) is an isometry having no unitary. By Wold's decomposition theorem [19, Th. I.1.1] we may write uniquely

\begin{equation}
D = \sum_{n=0}^{\infty} U^n N_\perp \quad N_\perp = D \oplus UD.
\end{equation}

Since for any \( m > 0 \)

\begin{align*}
U^{-m} D &= U^{-m} [D \oplus U^m D] \oplus U^m D \\
U^{-m} [(U^k N_\perp) \oplus U^m D] &= \\
\left( \sum_{k=-1}^{m-1} U^k N_\perp \right) \oplus D \\
\left( \sum_{k=0}^{m} U^k N_\perp \right) \oplus D
\end{align*}

we obtain by (2.2.1) that

\begin{equation}
\sum_{n=0}^{\infty} U^n N_\perp
\end{equation}

It follows that for arbitrary \( h \in H \)

\begin{equation}
h = \sum_{n=0}^{\infty} U^n P N_\perp U^n h, \quad \| h \|_H^2 = \\
\sum_{n=0}^{\infty} \| P N_\perp U^n h \|_H^2
\end{equation}

Hence the map

\begin{equation}
r_+: H \to \ell_2(\infty,\infty)
\end{equation}

defined by

\begin{equation}
r_+ h = (P N_\perp U^n h)_{n=-\infty}^{\infty}
\end{equation}
is isometric. Since for \( \{ h_n \}_{n=0}^{\infty} \in l_2 (\mathbb{R}^n; \mathcal{N}_+) \), \( h = \sum_{n=0}^{\infty} n^h \in \mathcal{H} \), the map \( r_+ \) is onto and thus unitary. By (2.2.3) we obtain (i). From (2.2.5)

\[
    r_+ U h = P_{\mathcal{N}_+} \left( U^{(n+1)} h \right)_{n=-\infty}^{\infty} = U_+ (r_+ h)
\]

and (ii) follows. By (2.2.4) \( U \) is a bilateral shift of multiplicity equal to \( \dim \mathcal{N}_+ \) and the uniqueness follows.

2.2.3 DEFINITION. The representation \( (U_+, l_2(0, \infty; \mathcal{N}_+); \mathcal{L}_2(\mathbb{R}^n; \mathcal{N}_+)) \) is called an outgoing transition representation.

For \( (U, \mathcal{L}_2, \mathcal{D}_+) \) incoming we similarly obtain

(2.2.6) \[
    \mathcal{D} = \sum_{n=0}^{\infty} U^n \mathcal{N}_+ , \quad \mathcal{N} = \mathcal{D} \oplus U^\star \mathcal{D}^\star
\]

and

(2.2.7) \[
    \mathcal{H} = \sum_{n=-\infty}^{\infty} U^n \mathcal{N}_+ .
\]

For the corresponding map \( r_- \) of \( \mathcal{H} \) onto \( l_2(-\infty, 0; \mathcal{N}_-) \) we define

(2.2.8) \[
    r_- h = (P_{\mathcal{N}_-} U^{(n+1)} h)_{n=-\infty}^{\infty} .
\]

Thus

(i) \[ r_- [\mathcal{D}] = l_2(-\infty, 0; \mathcal{N}_-) \]

(ii) \[ U_- = r_- U r_+^{-1} \]

is the right shift on \( l_2(-\infty, 0; \mathcal{N}_-) \). The representation \( (U_-, l_2(-\infty, 0; \mathcal{N}_-), l_2(-\infty, 0; \mathcal{N}_-)) \) is called an incoming translation representation.

2.2.4 DEFINITION ([11], [15]). The operator

\[
    S = r_+ r_+^{-1} : l_2(-\infty, 0; (\mathcal{C}^p)) \to l_2(-\infty, 0; (\mathcal{C}^p))
\]

is called the abstract scattering operator.

Clearly \( S \) is unitary. Denoting by \( V \) the right shift on \( l_2(-\infty, 0; (\mathcal{C}^p)) \), we readily obtain by the translation representation theorem

(2.2.9) \[ SV = r_+^{-1} r_+ V = r_- U r_+^{-1} V S . \]

Let \( F : l_2(-\infty, 0; (\mathcal{C}^p)) \to L_2(\mathcal{C}^p) \) be the Fourier transform operator. The unitary operator

\[
    F S F^{-1} : L^2(\mathcal{C}^p) \to L^2(\mathcal{C}^p)
\]

thus commutes by (2.2.9) with \( L_\chi \), the operator of multiplication by \( \chi \). It follows [19] that \( F S F^{-1} \) is a Laurent operator \( L_S \in L_\infty(B(\mathcal{C}^p)) \), such that

\[
    S(e^{ik}) \quad \text{a.e. is a unitary map on } \mathcal{C}^p .
\]

2.2.5 DEFINITION ([11], [15]). \( S \) is called the scattering matrix.

It is clear from the translation representation theorem that \( S \) is determined to within right and left multiplication by unitary transformations on \( \mathcal{C}^p \).

The unitary maps \( F_- = F r_- \), \( F_+ = F r_+ \), are called the incoming and outgoing spectral representation. We have the following:

a) \[ F_- (\mathcal{D}) = H^2(\mathcal{C}^p) . \]

(2.2.9) \[ F_+ (\mathcal{D}) = SH^2(\mathcal{C}^p) . \]

c) \[ F_- (U h) = \chi F h , h \in \mathcal{H} . \]

Moreover, the operator
(2.2.10) \( P_+ P_+^*: D_+ \to D_+ \) is unitarily equivalent to the Hankel operator \( H_0 \), where \( P_+ = P_{D_+} \), and the operator

\[
P D_+ \uplus P_+^*: D_+ \to D_+^{\uplus}
\]
is equivalent to the Toeplitz operator \( T_0 \).

(2.2.11) \( PD_+^{\uplus} P_+: D_+ \to D_+^{\uplus} \) is equivalent to the Toeplitz operator \( T_0 \).

2.3 COMPUTATION OF THE SCATTERING FUNCTION FOR REGULAR, MAXIMAL RANK, STATIONARY GAUSSIAN SEQUENCES

Let \((\Omega, \mathcal{A}, P)\) be a fixed probability space and let

\[
(y(n): n \in \mathbb{Z}), y(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \\ \vdots \\ y_p(n) \end{pmatrix}
\]

be a centered stationary process with \( y_j(n) \in L^2(\Omega, \mathcal{A}, P) \) \( j = 1, \ldots, p \). Let \( f_\gamma(\lambda) = \sum_{k=-\infty}^{\infty} f_{\gamma}(\lambda_k) \) be its spectral density satisfying

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det f_{\gamma}(\lambda) \, d\lambda > -\infty
\]

i.e., the process is regular and of maximal rank. Let

\[
H = \mathcal{H}_0 = V \{ y_1(n), y_2(n), \ldots, y_p(n) \} \subset L^2(\Omega, \mathcal{A}, P)
\]

be the space spanned by the process and let \( U \) be the unitary shift operator on \( H \) associated with the \( y \) process [21, p. 14]:

\[
U y_j(n) = y_j(n+1) \quad j = 1, \ldots, p, \quad n \in \mathbb{Z}
\]

We consider the past and future of \( \{y(n)\}_{n=\infty}^{\infty} \) defined by

\[
\begin{align*}
D_- = H_0^* (0) &= V \{ y_1(k), \ldots, y_p(k) \}_{k=0}^{\infty} \\
D_+ = H_0^+ (0) &= V \{ y_1(k), \ldots, y_p(k) \}_{k=0}^{\infty}
\end{align*}
\]

By (2.3.1) it follows [21, Th. II.6.1]

\[
\begin{align*}
\Lambda^\top U^n D_+ |_{n=0} &= \Lambda^\top U^n D_-
\end{align*}
\]

We readily obtain that \( (U, H, D_+, D_-) \) is a scattering system.

Now let \( (U, H, D_+, D_-) \) be the scattering system associated with the regular maximal rank \( y \) process. The subspace \( N = D_0 \oplus U^* D_+ (N = D_0 \oplus U^* D_-) \) is the forward (backward) innovation subspace at \( n = 0 \). Since for a scattering system \( (U, H, D_+, D_-) \) we have

\[
\dim N = \text{multiplicity } U = \dim N_+
\]

we can arrange the maps \( r_+ \) to be onto \( \mathbb{L}^{1, \infty} \). We next compute the scattering matrix \( S \) for the \( y \) process. Let \( \{v_1(0), \ldots, v_p(0)\} \) be an orthonormal basis for \( N_+ \). Let \( v_j(n) = U^j v_j(0) \) and define

\[
\chi(n) = \begin{pmatrix} v_1(n) \\ \vdots \\ v_p(n) \end{pmatrix}, \quad n \in \mathbb{Z}
\]

By (2.2.4) the process \( \{v(n)\}_{n=\infty}^{\infty} \) is a (centered) white noise process with covariance \( R_{v,n}(t) = \delta(t) I_p \), constituting the forward innovation process for the \( y \) process. It is determined up to a choice of basis in \( N_+ \). By (2.2.6) we may write
\[ y(0) = \sum_{k=-\infty}^{\infty} A(k)x(k) \quad A(k) = (a_{ij}(k))_{i,j=1}^{p} \quad A(0) = 0, \ k > 0. \]

(Wold's representation). It follows from (2.2.8)

\[ r_{y}(0) = \left\{ \sum_{m=1}^{p} \alpha_{m}(k+1)B_{m}(0) \right\}_{k=-\infty}^{\infty} \]

Identifying \( N_{-} \) with \( C^{0} \) we readily obtain the representation

\[ r_{y}(0) = \left( \begin{array}{c}
  \alpha_{1}(k+1) \\
  \vdots \\
  \alpha_{p}(k+1)
\end{array} \right)_{k=-\infty}^{\infty} \]

Consider the function

\[ \Lambda(z) = \sum_{k=-\infty}^{\infty} A^*(k)z^{k} \]

Since

\[ \sum_{k=-\infty}^{\infty} \sum_{i,j=1}^{p} |\alpha_{ij}(k)|^{2} \leq \sum_{j=1}^{p} \|y_{j}(0)\|^{2} \]

\( \Lambda(z) \) is analytic in \( |z| > 1 \). For \( \Lambda(z) \) we have,

\[ \frac{1}{2\pi} \Lambda^{*}(z)\Lambda(z) = \frac{f_{y}(\lambda)}{f_{y}^{*}(\lambda)} \]

By the incoming properties

\[ H_{2}(C^{0}) = V_{\infty} \{ e^{ikA} \alpha(k) : \alpha \in (C^{0}) \} \]

i.e., \( A \) is conjugate outer \([14, \text{p. 121}]\). Thus

\[ (Fr_{+}y_{1}(0), \ldots, Fr_{+}y_{p}(0)) = \overline{\Lambda} \]

Since the translates (in \( H_{2} \)) of \( y_{1}(0), \ldots, y_{p}(0) \) and their linear combinations are dense in \( H_{2} \). \( Fr_{+} \) is determined by the above expression.

We now consider the outgoing representation. Let \( \epsilon_{1}(0), \ldots, \epsilon_{p}(0) \) be an orthonormal basis in \( N^{+} \). We similarly obtain

\[ y(0) = \sum_{k=-\infty}^{\infty} B(k)x(k) \quad B(k) = (\beta_{ij}(k))_{i,j=1}^{p} \quad B(0) = 0, \ k < 0. \]

This representation constitutes the representation of \( y(0) \) in terms of the backward innovation process \( \{e_{n}(n)\}_{n=-\infty}^{\infty} \). \( e_{n}(n) = \sum_{k=0}^{n} e_{p}(k) \). We define

\[ \Gamma(z) = \sum_{k=0}^{\infty} B^*(k)z^{k} \]

which is analytic in \( |z| < 1 \). In a similar fashion we obtain by direct computation

\[ \frac{1}{2\pi} \Gamma^{*}(z)\Gamma(z) = \frac{f_{y}(\lambda)}{f_{y}^{*}(\lambda)}, \quad z = e^{ik} \]

with \( \Gamma \) being outer. Also

\[ (Fr_{+}y_{1}(0), \ldots, Fr_{+}y_{p}(0)) = \Gamma \]

Combining (2.3.3) with (2.3.4), we obtain

\[ S\Gamma = \overline{\Lambda} \]

and thus

\[ S = \overline{\Lambda} \Gamma^{-1} \]

One easily verifies that \( S(e^{ik}) \) is unitary a.e. \( \lambda \in [-\pi, \pi] \). We thus obtain

2.3.1 THEOREM. For a regular maximal rank process \( \{y(n)\} \) we have

\[ S = \overline{\Lambda} \Gamma^{-1} \]

where \( S \) is determined up to left and right multiplication by constant unitary matrices.

For the case \( p = 1 \) we have

2.3.2 COROLLARY. For a regular process \( \{y(n)\} \)
and $S$ is determined up to multiplication by a constant of unit modulus.

Proof. The outer function $\bar{\Lambda}$ satisfies $|\bar{\Lambda}| = \|\Gamma\|$ on $T$ and thus $\bar{\Lambda} = \gamma \Gamma$ a.e. where $\gamma$ is a constant of unit modulus.

2.3.3 Remark. The scattering matrix $S$ was defined by outer and conjugate outer factors of the density $f_{yy}$. Since those are determined up to left multiplication by a constant unitary matrix, we may wish to make a canonical choice (which amounts to choosing specific orthonormal bases in $N_+, N_-$) in the following fashion: For $\Gamma(0)$ we consider its polar decomposition $\Gamma(0) = KP$ ($K$ unitary, $P > 0$) and define $\Gamma_1(z) = K^{*-1}\Gamma(z)$. For $\Gamma_1$, we have $\Gamma_1(0) = \gamma$. This $\Gamma_1$ is unique. Similarly for $\Lambda$. In this way, the density $f_{yy}$ will have a unique $S$ associated with it. From the viewpoint of seeing $S$ as the phase function associated with $f_{yy}$, this may be appealing.

2.4 Completely Non-Deterministic Stationary Sequences, Their Scattering Functions, and Induced Hankel Operators

The scattering function of stationary sequences plays the role of the Heisenberg $S$-matrix in quantum mechanics. The physics of quantum systems is believed to be contained in the $S$-matrix and this object can in principle be determined experimentally. A natural question then is whether the scattering function of stationary Gaussian sequences, which measures the interaction between the past and future of the process, determines the spectral density of the process. To answer this question we introduce the class of completely non-deterministic processes.

2.4.1 Definition [7]. The process $y$ is said to be completely non-deterministic if

\begin{equation}
\Pi_2^+(0) \cap \Pi_2^+(1) = [0].
\end{equation}

This condition states that no value in $\Pi_2^+(1)$ can be predicted without error based on the information $\Pi_2^+(0)$. This condition is more restrictive than regularity (cf.

Bloomfield-Jewell-Hayashi, loc.cit. for an example of a regular process which is completely non-deterministic).

2.4.2 Theorem. The scattering matrix $S$ determines the spectral density $f_{yy}$ up to the form $K^* f_{yy} K$ where $K$ is a constant $p \times p$ non-singular matrix if $y$ is completely non-deterministic.

2.4.3 Remark. For $p=1$, this result was obtained by Levinson and McKean [16].

2.4.4 Lemma. The scattering matrix $S$ determines the density $f_{yy}(\lambda)$ up to the form

\begin{equation}
K^* f_{yy}(\lambda) K
\end{equation}

where $K$ is a constant $p \times p$ non-singular matrix, iff

\begin{equation}
\dim \text{Ker} T_S = p.
\end{equation}

Proof. First note that for any representation of $S$

$$S = \chi YX^{-1}$$

with the columns of $X$ in $L^2(C^2)$ and those of $\chi Y$ in $L^2(C^2)$, the columns of $X$ belong to Ker $T_S$. Moreover (on $T$) $f_y$

$$Y^* Y = (SX)^* SX = X^* X$$

Assume (2.4.3) holds. It thus follows that

\begin{equation}
X(e^{i\lambda}) = \Gamma(e^{i\lambda}) K
\end{equation}

where $K$ is a $p \times p$ full rank constant matrix. Thus,

$$\frac{1}{2\pi} X^*(z) X(z) = \frac{1}{2\pi} K^* f_{yy}^*(z) \Gamma(z) K = K^* f_{yy}(\lambda) K \quad z = e^{i\lambda}$$
proving the if part.

Now assume (2.4.3) not to hold, i.e., \( \dim \ker T_S > p \). We can thus find a \( p \times p \) matrix \( \chi X(\epsilon) \) of full rank a.e. \( \lambda \) such that the columns of \( X \) belong to \( \ker T_S \) and (2.4.4) does not hold. If we define

\[ Y = \tilde{\chi} SX \]

then the columns of \( \tilde{\chi} Y \) are in \( \tilde{H}_2^+ (C^0) \) and \( S = \tilde{\chi} Y X^{-1} \) with \( Y^* Y = X^* X \). The result follows.

We next characterise condition (2.4.3) on a process level.

2.4.5 Lemma. We have

\[ F_+^* [\ker T_S] = H_2^+ (0) \Lambda H_2^+ (0). \]

Proof. Let \( 0 \neq f \in \ker T_S \). From the well-known identity \( H_S^* H_S + T_S^* T_S = 1 \), it follows that

\[ H_S^* H_S f = f \]

i.e.,

\[ \| H_S f \| = \| f \|. \]

From (2.2.10) and (2.2.11) we obtain for \( \xi = F_+^* f \in H_2^+ (0) \)

\[ H_2^+ (0) \Lambda H_2^+ (0). \]

and \( \xi \in H_2^+ (0) \). Thus

\[ F_+^* [\ker T_S] \subset H_2^+ (0) \Lambda H_2^+ (0). \]

Now let \( \xi \in H_2^+ (0) \Lambda H_2^+ (0) \). It follows from (2.2.10) and (2.2.11)

\[ H_S (F_+^* \xi) = F_+^* \xi. \]

Let \( f = F_+^* \xi \in H_2^+ (0) \). We obtain

\[ \| H_S f \| = \| F_+^* \xi \| = \| H_2^+ (0) \Lambda H_2^+ (0) \| = \| f \|. \]

and

\[ H_S^* H_S f = f. \]

Thus \( f \in \ker T_S \) which implies

\[ F_+ [H_2^+ (0) \Lambda H_2^+ (0)] \subset \ker T_S. \]

By the unitarity of \( F_+ \)

\[ \dim \ker T_S = \dim H_2^+ (0) \Lambda H_2^+ (0), \]

and since \( y \) is regular and of full rank we readily conclude

\[ \dim H_2^+ (0) \Lambda H_2^+ (0) = p \quad \text{i.e.} \quad \dim H_2^+ (0) \Lambda H_2^+ (1) = 0. \]

Proof of Theorem 2.4.2. Combine Lemmas 2.4.4. and 2.4.5.

The converse question, namely, when is a function \( S \in L_\infty (B(C^0)) \) the scattering matrix of some full rank, \( p \)-dimensional completely non-deterministic process is of interest.

We first observe that any \( S \in L_\infty (B(C^0)) \) which is unitary a.e. on \( T \) is the scattering matrix of the canonical scattering system [1]

\[ U = L_2 (\mathbb{C}), \quad H = L_2 (\mathbb{C}^0), \quad D_+ = SH_2 (\mathbb{C}^0), \quad D_- = H_2 (\mathbb{C}^0). \]

The above question amounts to characterizing all scattering systems \((U, H, D_+, D_-)\) for which there exists a set \( \{ \xi_j : j=1, \ldots, p \} \) of linearly independent vectors such that

\[ H = \text{span} \{ U^n \xi_j : j=1, \ldots, p, n=0, \pm 1, \ldots \} \]
\[ D_+ = \text{span} \left\{ U^n \xi_j : j = 1, \ldots, p, \ n \geq 0 \right\} \]

\[ D_- = \text{span} \left\{ U^n \xi_j : j = 1, \ldots, p, \ n \leq 0 \right\} \]

and such that any other linearity independent set satisfying the above is of cardinality \( p \). The corresponding process will be \( \{ \xi(n) \}_{n=0}^{\infty} \) where \( \xi(0) = \left( \begin{array}{c} \xi_1 \\ \vdots \\ \xi_p \end{array} \right) \)

\[ \xi(n) = \begin{pmatrix} U^n \xi_1 \\ \vdots \\ U^n \xi_p \end{pmatrix}, \]

and the spectral density is obtained by

\[ f_{\xi \xi}(\lambda) = \frac{d\langle \xi, \xi \rangle}{d\lambda} \] for \( U \). The answer is given in the following.

\[ \text{2.4.6 THEOREM. Let } S \in L_\infty(B(B^p)) \text{ be such that} \]

(i) \( S(e^{i\lambda}) \) is a.e. \( \lambda \) a unitary map on \( (B^p) \)

(ii) \( \dim \ker T_S = p \)

Then there exists a \( p \)-dimensional full rank completely non-deterministic process \( y \) whose scattering matrix is \( S \).

\[ \text{Proof. Let } \Gamma_1, \Gamma_2, \ldots, \Gamma_p \text{ span the kernel of } T_S \text{ and define} \]

\[ \Gamma = [\Gamma_1 \ | \Gamma_2 \ | \ldots \ | \Gamma_p] \]

Let

\[ \Lambda = S\Gamma. \]

Since \( \Lambda_j = (S\Gamma_j) + \pi_+(S\Gamma_j) = \pi_+(S\Gamma_j), j = 1, \ldots, p \), the columns of \( \Lambda = [\Lambda_1 \ | \Lambda_2 \ | \ldots \ | \Lambda_p] \) are in \( H_2^p (C^p) \) and by (i)

\[ \Lambda^*(z)\Lambda(z) = \Gamma^*(z)\Gamma(z), \quad z = e^{i\lambda}. \]

If we define

\[ f_{\Lambda^*}(\lambda) = \frac{1}{2\pi} \Gamma^*(e^{i\lambda})\Gamma(e^{i\lambda}) \]

the theorem follows provided we show that \( \Gamma \) is outer and \( x \Lambda \) conjugate outer. Let \( U = L_x \) and define:

\[ \hat{D}_- = \mathcal{V}_{n \in \mathbb{N}} [\chi^n \Lambda_1, \ldots, \Lambda_p] \subset H_2^p (C^p), \]

\[ \hat{D}_+ = \mathcal{V}_{n \in \mathbb{Z}} [\chi^n S\Gamma_1, \ldots, \chi^n S\Gamma_p] \subset SH_2^p (C^p). \]

Let

\[ (2.4.5) \quad \mathcal{H} = (\mathcal{V}_{n \in \mathbb{Z}} U^n \hat{D}_-) \mathcal{V} (\mathcal{V}_{n \in \mathbb{Z}} U^n \hat{D}_+). \]

It is easily verified that \( (2.2.1)_\pm - (i), (ii) \) holds for \( (U, D_\pm) \). In [1, Adamjan-Arov show [1, Th. 2.5] that a quadruple \( (U, H, D_+, D_-) \) satisfying \( (2.2.1)_\pm - (i), (ii) \) and \( (2.4.5) \) has a scattering matrix \( S \) which is unitary valued a.e. on \( T \) iff

\[ \mathcal{V}_{n \in \mathbb{Z}} U^n D_+ = H = \mathcal{V}_{n \in \mathbb{Z}} U^n D_- \]

and, moreover, from their generalized functional model [1, Th. 2.1] we need have

\[ \hat{D}_- = H_2^p (C^p), \hat{D}_+ = \hat{S} H_2^p (C^p). \]

A straightforward computation gives:

\[ \hat{S} = S \]

and the result follows.
From the theorem of Nehari [3], and its vector generalization we know that for a bounded Hankel operator $H$ with symbol $\varphi \in L_\infty$, there exists a function $\varphi_\mu \in L_\infty$ such that

$$\|H\|_\mu = \|\varphi_\mu\|_\infty.$$

The function $\varphi_\mu$ is called a minification for $H$. In general $\varphi_\mu$ is not unique. We however have

2.4.7 THEOREM. The Hankel operator $H_S$ determines $S$ uniquely. Indeed, $S$ is its unique minification.

Proof. From Lemma 2.4.4 we note that $f \in \ker T_S$ is an eigenvector of $H_S^* H_S$ corresponding to the eigenvalue $\|H_S\|_1 = 1$. Since $S = \chi A^{-1}$ every column of $\Gamma$ belongs to this kernel. Thus, the projection of the above eigenspace on the first coordinate in $L_2(0,\infty; C^m)$ spans $\Gamma(0)$. Now observe that for $\Gamma(0)$ we have, because of its outer property in $L_2(B(C^m))$,

$$\log \frac{\log \det \Gamma(0)}{(2\pi)^{n/2}} = \frac{1}{4\pi} \int_{-\pi}^\pi \log \det f_{\lambda}(\lambda) d\lambda > -\infty,$$

so that $\Gamma(0)$ is of full rank. We conclude that the aforementioned projection is onto the first coordinate space. According to a result of Adamjan-Arov-Krein [2, Corollary 3.1] for Hankel operator $H_S$ to have a unique minification, it is sufficient that the projection of the eigenspace of $H_S^* H_S$ corresponding to $\|H_S\|_1$ on the first coordinate space be onto. The result follows.

2.4.8 REMARK. It is of interest to observe that since for a completely nondeterministic process, the eigenvectors of $H_S^* H_S$ corresponding to the eigenvalue $\|H_S\|_1$ are only the columns of $\Gamma$, the projection of this eigenspace on the first coordinate is not only onto, but also $1$-$1$. In [2, Sec. 2] it is shown that for any Hankel operator $H : H_2(C^m) \to H_2(C^m)$ satisfying this condition its unique minification is of the form $\rho S$, $\rho = \|H\|_1$.

Thus, up to a constant multiple $\rho > 0$ all minifications of such Hankel operators are in $1$-$1$ correspondence with regular, full rank, completely nondeterministic processes.

The case of Rational Functions. Let $(y(n))_{-\infty}^\infty$ have rational density

$$f_Y(\lambda) = \frac{|P(z)|^2}{|Q(z)|^2},$$

where the polynomials $P, Q$ have no zeros in $|z| < 1$ and are relatively prime. Since $f_Y \in L_1$, the polynomial $Q$ has its zeros in $|z| > 1$. Write

$$P = P_1 P_2$$

where $P_1$ of degree $k$ has its zeros on $T$ and $P_2$ in $|z| > 1$. For $P_1(z) = \prod_{j=1}^k (z - \alpha_j)$, $(\alpha_j)_{j=1}^k \subset T$ we have

$$\frac{P_1(e^{i\lambda})}{P_1(e^{i\lambda})} = e^{-i\lambda k} (-1)^k \prod_{j=1}^k \tilde{a}_j.$$

Thus

$$(2.4.6) \quad S = \gamma^{-k+1} \frac{\psi_c}{\psi_e} \quad \psi_e = \frac{P_2}{Q}, \quad \gamma = (-1)^k \prod_{j=1}^k \tilde{a}_j$$

where $|\gamma| = 1$ and $\psi_e$ is outer. In [3] Adamjan-Arov-Krein show that (2.4.6) is the general form of unimodular minifications and that in this case $k+1$ is the dimension of the eigenspace corresponding to the singular value $1 = \|H_S\|_1$. We conclude that a regular process with rational spectral density is completely nondeterministic if it has no zeros on $T$.

2.5 MARKOV PROCESSES AND UNITARY DILATIONS

In this section we show how a Markov structure is intrinsically associated with unitary dilations and the resulting Scattering theory of Lax and Phillips. The results in this section are due to Lindquist and Picci [18] and Foias and Fraaho
2.5.1 PROPOSITION. $X \subset H_Y$ is a Markov subspace of full range iff $U (on \ H_Y)$ is the minimal unitary (power) dilation of the state operator

$$A = P_XUX : X \rightarrow X$$

Proof. From (2.5.1) we obtain for $x, x' \in X$ and $m, n \geq 0$

$$(U^m x, U^n x') = (U^m x, P_X U^n x').$$

Denoting $A(n) = P_X U^n I_X$, we obtain

$$(x, A(m+n)x') = (x, U^{m+n} x') = (U^m x, U^n A(n)x') = (x, P_X U^m P_X U^n x') = (x, A(m)A(n)x').$$

We infer that $A(m+n) = A(m)A(n)$ and

$$A(n) = A_n(1) = A^n.$$

Since $X$ is of full range, we conclude that $U$ in $H_Y$ is the minimal unitary dilation of $A (in X)$. This proves the 'only if' part. The 'if' part follows by reversing the argument.

Having made the connection between a Markov process $(U^n X)$ and the dilation property characterizing it, the work of Adamjan-Arov [11] on the duality between dilation theory and the scattering operator model is directly applied. First, note that the process $(U^n X)$ is regular, i.e., satisfies

$$\forall \ n \geq 0 \ k \geq n \ A_n \forall U^kX = \{0\} = A \forall U^kX$$

iff

$$A^n \rightarrow 0 \ \text{and} \ \ A^n \rightarrow 0 \ (n \rightarrow \infty).$$

In a Hilbert space setting, a centered stationary process $\{x(n)\}_{-\infty}^{\infty}$ is said to be Markov if for all $n \geq s$

$$P_h \mathcal{H}_X(s) X(s) = h \in \mathcal{H}_X(n),$$

where $X(s) = \text{span} \{x_j(s) : j = 1, \ldots, m\}$. In our setting, all stationary processes will be generated by the shift $U$ (on $H_Y$) associated with the $y$ process. Thus, for a stationary process $\{x(n)\}_{-\infty}^{\infty}$ (in $H_Y$) we will have $x(n) = U^n x(0)$. It readily follows from above that one can define the notion of a Markov subspace $X \subset H_Y$ (for $U$) if $X$ satisfies (see [18])

$$(2.5.1) \quad P_{X, x} U^n x = P_{x, x} U^n x, \quad n \geq s, x \in X.$$

Thus $X$ is a Markov subspace (for $U$) iff the process $(U^n X)$ has the (weak) Markov property. In what follows a Markov process $(U^n X)$ will invariably arise in this fashion.

Markov subspaces $X \subset H_Y$ which are representations for the process $y$, i.e., for which

$$[y_1(0), \ldots, y_p(0)] \subset X,$$

satisfy

$$H_Y = \bigoplus_{n=-\infty}^{\infty} U^n X,$$

and are said to be of full range. There is a direct relationship between Markov processes of full range and unitary dilations (see also [10]). Recall [19] that a unitary operator $U$ on a Hilbert space $H$ is said to be the minimal unitary (power) dilation of a contraction $A$ on $X \subset H$ if

$$A^n = P_X U^n I_X \quad n \geq 0 \ \text{and} \ \ H = \bigoplus_{n=-\infty}^{\infty} U^n X \ (\text{minimality}).$$
Second, those Markov processes which in addition to being regular represent $y$ (and are thus of full range) correspond to scattering systems according to a result of Adamjan-Arov [1, Th. 3.4]:

2.5.2 THEOREM. Let $X \subset H^2_0$ be a regular Markov subspace of full range. Then $H^2_0$ decomposes and, moreover, uniquely into the orthogonal sum

$$H^2_0 = D \otimes X \oplus D_\perp,$$

where $(U, H^2_0, D, D_\perp)$ is a scattering system.

2.5.3 DEFINITION. A scattering system $(U, H^2_0, D, D_\perp)$ for which

$$D_\perp \subset D$$

is called a Lax-Phillips (L-P) scattering system.

Let $(U^N X)$ be an arbitrary regular Markov process of full range, and $(U, H^2_0, D, D_\perp)$ its associated L-P scattering system. Let $\Theta_X(e^{i\lambda})$ be the corresponding scattering matrix. For the induced incoming spectral representation $F^-_X$ we obtain

$$F^-_X [D] = H^2_0 \langle C^0 \rangle; \quad F^-_X [D_\perp] = \Theta_X [H^2_0 \langle C^0 \rangle].$$

Since $D_\perp \perp D$

$$\Theta_X \in H^\infty (\mathbb{C}^n) \langle C^0 \rangle.$$

To each regular full range Markov process there is thus associated an inner function $\Theta_X$, which is the scattering matrix of the corresponding L-P system $(U, H^2_0, D, D_\perp)$. From [1, Th. 3.3] it follows that the scattering matrices $\Theta$ associated with regular full range Markov processes are precisely the inner functions $\Theta \in H^\infty (\mathbb{B} \langle C^0 \rangle)$ which are purely contractive [19, p. 188], i.e.; for which

$$\|\Theta(0)\| < 1.$$

2.6 FACTORIZATION OF THE SCATTERING MATRIX AND MODELLING

The 1-1 correspondence

$$X \leftrightarrow (U, H^2_0, D, D_\perp)_X,$$

enables us to translate the realization problem of finding all regular Markovian representations for $y$ to a covering problem in $L^2_2(\mathbb{C}^n)$ via the outgoing spectral representation for $(U, H^2_0, D, D_\perp)$. Let $X \subset H^2_0$ be a regular Markov subspace representing $y$, $(U, H^2_0, D, D_\perp)_X$ its L-P scattering system, $\Theta X$ its scattering matrix, and $F^+_X$ the corresponding outgoing spectral representation. Since $(y(0), ..., y(p(0))) \subset X$ it follows

$$F^+_X [H^2_0 (0)] \in F^+_X \left[ \bigoplus_{k \leq n} U^k X \right] = F^+_X [D, \Theta X] = \mathbb{H}^2_0 \langle C^0 \rangle,$$

and $F^+_X [H^2_0 (0)]$ is a full range left shift invariant subspace of $\mathbb{H}^2_0 \langle C^0 \rangle$. Let $V$ be the corresponding inner function obtained from the Beurling-Lax theorem, i.e.,

$$F^+_X [H^2_0 (0)] = V^* \mathbb{H}^2_0 \langle C^0 \rangle.$$

Be the unitary of $F^+_X$

$$F^+_X (y(0)) = \overline{\Lambda} V^* \Lambda.$$

Thus

$$(y(0), ..., y(p(0))) \subset X.$$
translate under \( F_X^* \) to the equivalent condition

\[
(2.6.2) \quad \bar{\gamma} V^* \Lambda \in H_2^0(C^n) \otimes \Theta^* H_2^0(C^n).
\]

Conversely, if \( \Theta, V \) are inner functions for which (2.6.2) holds, the mapping

\[
y(0) \to \bar{\gamma} V^* \Lambda
\]

induces in a natural fashion a spectral representation \( F_{\Theta, V} \) for which

\[
X = F_{\Theta, V}^{-1} \left| H_2^0(C^n) \otimes \Theta^* H_2^0(C^n) \right|
\]

is a Markovian representation for \( y \), and its corresponding \( L-P \) scattering system has its scattering matrix \( \Theta_X \) coinciding with \( \Theta \). We have therefore proved the following

2.6.1 THEOREM. Finding all models (realizations) of \( y \) is equivalent to finding all inner functions \( \Theta_1 \) such that

\[
\bar{\gamma} V^* \Lambda \in H_2^0(C^n) \otimes \Theta_1^* H_2^0(C^n)
\]

for some inner function \( V \). Each model \( \Theta, V \) corresponds to a pair \( (\Theta_1, V) \).

2.6.2 COROLLARY. All regular Markov subspaces \( X \subset H_2^0(C^n) \) representing \( y \) are parametrized by those and only those inner functions \( \Theta_1 \) for which

\[
(2.6.3) \quad V^* S = \Theta_1^* \Theta_2 \quad \Theta_2 \in \mathcal{H}_\infty(B(C^n))
\]

for some inner function \( V \).

Proof. (2.6.2) holds iff \( \Theta_1 V^* \Lambda \in H_2^0(C^n) \) iff \( \Theta_1 V^* S \Lambda \in H_2^0(C^n) \). Since \( \Gamma \) is outer the latter holds iff \( \Theta V^* S \in \mathcal{H}_\infty(B(C^n)) \), i.e., iff (0.3) holds.

2.6.3 COROLLARY. All regular Markov subspaces \( X \subset H_2^0(C^n) \) representing \( y \) are parametrized by those and only those inner functions \( \Theta_1 \) for which

\[
(2.6.4) \quad S = \Theta_1^* \Theta_2 \quad \Theta_2 \in \mathcal{H}_\infty(B(C^n))
\]

\[
(2.6.5) \quad H_2^0(C^n) \otimes H_2^0(B(C^n)) \quad \text{is of full range for } U.
\]

A \( L_\infty(B(C^n)) \) function satisfying (2.6.4) is called [9] strictly non-cyclic, and the corresponding process - having a strictly non-cyclic scattering matrix - is called strictly monicic. We thus obtain [18, Lemma 7.3 and Th. 7.6]:

2.6.4 COROLLARY. Let

\[
S = Q_1^* Q_2 = P_2 P_1^*
\]

be respectively the left, right coprime factorization of \( S \). Then all minimal regular Markov subspaces representing \( y \) are parametrized by those and only those inner functions \( \Theta_1 \) such that

\[
(2.6.6) \quad V^* S = \Theta_1^* \Theta_2
\]

where \( \Theta_1, \Theta_2 \) are left coprime and \( V \) is an arbitrary left divisor of \( P_2 \). Moreover we have
\[ \text{det } \Theta_1 = \text{det } Q_1. \]

**Proof.** Combine Corollary (2.6.2) with [14, Lemma III. 5-8].

The general Fuhrmann degree theory for strictly non-cyclic functions [12, Ch. iii.5] now arises naturally - S playing a central role. All regular Markovian subspaces \( X \subset H \) representing \( y \) are parametrized by an inner function \( \Theta_1 \in H_{\infty}(B(C^0)) \) will be of degree
\[ d(\Theta_1) = \text{det } \Theta_1, \]

an inner function in \( H_{\infty} \), and
\[ d(Q_1) \text{ divides } d(\Theta_1). \]

Thus, the degree of the minimal subspace is the lowest, in the sense that \( d(Q_1) \) is the weakest among the degrees of all other regular Markovian subspaces representing \( y \). Applying [12, Th. II. 14.11] we infer that two minimal regular Markov subspaces representing \( y \) are quasi-similar.

2.6.5 COROLLARY. For a \( p = 1 \) dimensional process \( y \) the minimal Markovian representation of \( y \) is parametrized by the inner function \( q_1 \), for which
\[ s = q_1 q_2 \]

is a coprime factorization.

### III. UNITARY DILATIONS OF IRREVERSIBLE EVOLUTIONS

In the previous section we have shown how we can associate a Markov semigroup with a Gaussian process via the associated Scattering System. In this section, we consider the dual view of associating a Unitary Dilation with a Markov semigroup.

The evolution of a Hamiltonian (conservative) system is reversible while the evolution of a real physical system is not. The real system returns to a state of thermal equilibrium at a temperature determined by its surroundings. This is the physical interpretation of the Poincaré Recurrence Theorem. We may argue that the construction of a unitary dilation of a Markov semigroup is the abstract interpretation of coupling a physical system of a finite number of degrees of freedom to a heat bath thereby producing a Hamiltonian system of infinite number of degrees of freedom. The ideas of this section are due to Ford, Kac, Mazur [11], Lewis and Thomas [17], Evans and Lewis [9] and the first author.

We undertake the development of this section in continuous time since this is its natural setting.

#### 3.1 NOTATION AND PRELIMINARIES

Let \( H \) be a separable Hilbert space with scalar product, \( < \cdot, \cdot >_H \) and norm \( \| \cdot \|_H \). When there is no confusion the subscript \( H \) will be dropped. \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^+ \) the non-negative real numbers and \( \mathbb{R}^* \) the nonpositive real numbers. For \( I \subset \mathbb{R} \), an interval, \( L^2(I) \) denotes the space of Borel measurable square-integrable complex-valued functions and \( L^2(1; H) \) the space of Borel-measurable \( H \)-valued square integrable functions. \( W^{1,2}(R; H) \) denotes the Sobolev space of \( H \)-valued functions and \( W^{1,2}(I; H) \) the Sobolev space obtained by restriction of \( W^{1,2}(R; H) \). Let \( (T_t)_{t \in \mathbb{R}^+} \) denote a one-parameter, strongly continuous, contractive semigroup on \( H \), with \( T_0 = I \), and let \( A \) be the infinitesimal generator of \( (T_t)_{t \in \mathbb{R}^+} \). We know \( T_t = e^{-tA} \) is a contractive semi-group iff, \( \forall x \in D(A), 3 x^* \in H \) with \( \| x^* \| = 1 \), \( (x^*, x) = \| x \| \| x^* \| \) and \( \text{Re } <x^*, Ax> \geq 0 \).

A semigroup of contractions \( (T_t)_{t \in \mathbb{R}^+} \) is said to contract strongly to zero if \( \forall h \in H \), we have \( \lim_{t \to \infty} \| T_t h \| = 0 \).

#### 3.2 MINIMAL UNITARY DILATION OF A CONTRACTIVE SEMIGROUP

3.2.1 DEFINITION. Given a contractive semi-group \( (T_t)_{t \in \mathbb{R}^+} \) on \( H \), we say that a strongly continuous one-parameter group \( (U_t)_{t \in \mathbb{R}} \) on \( K \) is a unitary dilation of \( (T_t, H) \) if there exists an isometry \( i: H \to K \) such that the following diagram commutes:

The unitary dilation is said to be minimal if...
3.2.2 THEOREM [LAX-PHILLIPS]. Let \((T_t, \mathcal{H})\) be a strongly continuous, contractive, semigroup contracting strongly to zero. Then there exists a unitary dilation \((U_t, \mathcal{H})\). The dilation \((U_t, \mathcal{H})\) has the canonical representation with \(\mathcal{H} = L^2(\mathbb{R}; \mathbb{N})\), \(\mathbb{N}\) a Hilbert space and \((U_t)_{t \in \mathbb{R}}\) being the unitary group of right translations on \(L^2(\mathbb{R}; \mathbb{N})\):

\[
(U_t f)(s) = f(s-t).
\]

\((3.2.1)\)

Proof. Since \((T_t)_{t \in \mathbb{R}^+}\) is a contraction,

\[
Q(h) = \langle Ah, h \rangle + \langle h, Ah \rangle \geq 0 \forall h \in D(A).
\]

Let \(N_0 = \text{Ker}[Q(h)]\) and let \(P\) be the canonical projection of \(D(A)\) onto the quotient space \(D(A)/N_0\). On \(D(A)/N_0\) there exists a scalar product \(\langle \cdot , \cdot \rangle_A\) such that

\[
\langle Ph, Pk \rangle_A = \langle Ah, k \rangle + \langle k, Ah \rangle \quad \forall h, k \in D(A).
\]

Let \(N\) denote the Hilbert space completion of \(D(A)/N_0\) with respect to the norm induced by \((3.2.3)\). Therefore

\[
\int_0^t \| PT_s h \|^2_A ds = \| h \|^2_A - \| T_0 h \|^2_A, \quad \forall h \in D(A) \quad t \geq 0.
\]

\((3.2.4)\)

If we let \(t \to \infty\), since \(T_t\) contracts strongly to zero, there exists an isometric embedding \(i: \mathcal{H} \to L^2(\mathbb{R}; \mathbb{N})\), such that on \(D(A)\),

\[
(ih)(s) = PT_s h, \quad \forall s \leq 0.
\]

Regarding \(L^2(\mathbb{R}^+; \mathbb{N})\) as a subspace of \(L^2(\mathbb{R}; \mathbb{N})\), we have for \(\forall h \in D(A)\) and \(t \geq 0\)

\[
(U_t ih)(s) = \begin{cases} 
PT_s h & s \leq t \\
0 & s > t
\end{cases}
\]

\[(i T_t h)(s) = \eta_t(s)\]

where \(\eta_t(s) \in L^2(\mathbb{R}^+; \mathbb{N}) \subset i(H)^1\). Hence, \(\forall t \leq 0\)

\[
T_t = i^\ast U_t i, \quad \text{and therefore} \quad U_t \text{ is a unitary dilation of } T_t \text{ on } \mathcal{H} = L^2(\mathbb{R}; \mathbb{N}).
\]

The unitary dilation we have constructed is in fact minimal. This is done by constructing a linear stochastic differential equation involving an operator-valued Brownian motion. We first introduce positive definite kernels and consider their decomposition.

\[
\sum_{i,j=1}^n \langle K(x_i, x_j) h_i, h_i \rangle \geq 0.
\]

\((3.2.5)\)

3.2.3 DEFINITION. A map \(K: \mathbb{R} \times \mathbb{R} \to \mathcal{B}(\mathcal{H})\) is said to be a kernel. The set of all such maps is denoted by \(K(\mathbb{R}; \mathcal{H})\). A kernel \(K\) is said to be positive definite if \(\forall h_1, ..., h_n \in \mathcal{H}\) and \(x_1, ..., x_n \in \mathbb{R}\),

3.2.4 DEFINITION. Let \(K \in K(\mathbb{R}; \mathcal{H})\). Let \(\Pi\) be a Hilbert space and let \(V: \mathbb{R} \to \mathcal{B}(\Pi; \Pi)\) be such that \(K(x, y) = V(x) \ast V(y)\). Then \(V\) is said to be a Kolmogorov decomposition of \(K\). This decomposition is minimal if \(\Pi = \cup \{ V(x) h \mid h \in \mathcal{H} \}, \quad x \in \mathbb{R}\).

One can prove that every positive definite kernel has a minimal Kolmogorov decomposition. This is done with the aid of the reproducing kernel Hilbert space associated with \(K\).

With the notation of Theorem 3.2.2, let us introduce an operator-valued Brownian motion as follows:

Let \(W: \mathbb{R} \to \mathcal{B}(\mathbb{N}; \mathbb{H})\) be the map given by

\[
W_t(s, \eta)(s) = \begin{cases} 
\chi_{[0,t]}(s) \eta, & t \geq 0 \\
-\chi_{[t,0]}(s) \eta, & t < 0.
\end{cases}
\]

\((3.2.6)\)

where \(\eta \in \mathcal{N}\) and \(\chi(.)\) is the characteristic function.

Consider the positive-definite kernel:

\[
(s, t) \to (s \land t) \mathbb{N}, \quad \text{where } \mathbb{I} \text{ denotes the identity operator. Then}
\]

\[
(s, t) \mathbb{N} = W_t^* W_s.
\]

In the sequel we denote by \((D(A), I)\) the Hilbert space \(D(A)\), with the graph norm.
3.2.5 Theorem. Let \((U_t, \mathcal{H})\) be the dilation of \((T_t, H)\) given in Theorem 3.2.2. Then there exists a bounded linear operator \(B : (D(A^{1/2}), L^2) \to N\) and an operator-valued Brownian motion \(W_t : R \to \mathcal{B}(N; M)\), where
\[
M = \bigcup \{W_{t,n}s \in R, n \in N\} \text{ and } W_t \text{ satisfies (3.2.6) such that}
\]
\[
(3.2.7) \quad (U_t^{-1} - U_s)^{h} = \int \frac{1}{2} U_t \text{d}h + (W_t - W_s)B_h, \forall h \in D(A).
\]

Proof. The proof is constructed by verifying equation (3.2.7) for \(h \in D(A^2)\) and then by density for \(h \in D(A)\). For \(h \in D(A^2)\) one can show that a solution is given by
\[
U_t^{h} = e^{A(t-s)h}U_s^{h} + \int \frac{1}{2} W(\text{d}t)B e^{-A(t-s)}h,
\]
where the last term is a Wiener integral, which can be defined by an integration by parts formula. The fact that \(U_t\) is a minimal unitary dilation follows from the fact that \(w_t\) is a minimal Kolmogorov decomposition.

3.2.6 Remark. The stationary solution of the equation is given by
\[
U_t^{h} = \int W(\text{d}s)B e^{-A(t-s)}h.
\]
We may verify that this \(U_t\) defines a regular stationary Gaussian process and there is a Lax-Phillips structure associated with it. We may also obtain an ordinary stochastic differential equation for the Markov semigroup attached to this Lax-Phillips system.

3.2.7 A New Representation of the dilation.

Let us assume that the semigroup \((T_t)_{t \in R^+}\) on \(H\) is self-adjoint with generator \(-A\). Then \(A\) is a positive self-adjoint operator which we assume to be injective. In this case, one can show that there exists a minimal unitary dilation \((U_t, \mathcal{H})\), where \(H = H_1 \oplus L^2(R; H_2)\).

Let us write a vector \(\varphi \in L^2(R; H)\) as \(\varphi = \varphi^* + \varphi'\) with \(\varphi^* \in L^2(R^+; H)\) and \(\varphi' \in L^2(R^-; H)\). Then one can write the unitary dilation for \(t \in R^+\) as
\[
(3.2.8) \quad U_t = \begin{pmatrix} T_t & A_t \\ B_t & S_t + C_t \end{pmatrix},
\]
where
\[
\mathcal{A} : L^2(R; D(A^{1/2})) \to H; \varphi \to (2A)^{1/2} \int_0^t T_{t-s} \varphi(s) \text{d}s
\]
\[
B_t : D(A^{1/2}) \to L^2(R; H); h \to (B_t h)(s) = \chi_{[0,t]}(s) (2A)^{1/2} T_{t-s} h
\]
\[
C_t : L^2(R; D(A)) \to L^2(R; H); \varphi \to (C_t \varphi)(s) = \chi_{[0,t]}(s) (2A)^{1/2} A_{t-s} \varphi
\]
\[
S_t : L^2(R; H) \to L^2(R; H); \varphi \to (S_t \varphi)(s) = \varphi(s-t)
\]
\[
\mathcal{A}_t, B_t, C_t, S_t \text{ are densely defined contractions.}
\]
\[
\mathbf{K}_R \text{ the Hamiltonian of the reservoir is the generator of the shift of motion. } \mathbf{K}_s, \text{ the Hamiltonian of the system is zero. } \mathbf{K}_c , \text{ the Hamilton coupling is of the form}
\]
\[
(3.2.9) \quad \mathbf{K}_c = \begin{pmatrix} 0 & -iC^* \\ -C & 0 \end{pmatrix}
\]
where \(C : D(A^{1/2}) \to L^2(R; H); h \to \delta_0 \oplus (2A)^{1/2} h, \delta_0\) being the Dirac \(m\) formal and needs to be justified) and
\[
C^* : L^2(R; H) \to D(A^{1/2}); \varphi \to (2A)^{1/2} \varphi(0)
\]
(this is also formal).

We now give some indications on how the new construction is arr. Since \((T_t)_{t \in R^+}\) is contractive, the quadratic form \(F(x,x) = (Ax, x) + \lim_{\|\xi\| \to 0} \frac{\|T_t \xi\|^2 - 2T_{t} T_{t} \xi \|}{t} \leq 0 \) for \(x \in D(A)\). We claim that there exists an operator \((T_t, H_1) \times \mathcal{H} \times \mathcal{H}_2\) such that \(\mathbf{C}(x) \mathcal{H}_1 \times \mathcal{H}_2\).

In a similar manner there exists a pair \((C, H_2)\) for \(x \in \mathcal{H}_1\). The operators \(C\) and \(C^*\) are to be the coupling operators. In the self-adjoint case \(C = (2A)^{1/2}\).

The idea is to construct the dilation on the space \(H = L^2(R; H_1) \oplus \mathcal{H}_2\). Now since \((T_t, H)\) is a Markov semigroup, we must have \((U_t = P_{t}^{L})\)
orthogonal to i(H) where i : H → H is the injection \( \xi \rightarrow \begin{pmatrix} 0 \\ \xi \\ 0 \end{pmatrix} \), \( P_0 : H \rightarrow H \) is the orthogonal projection and \( P_0^* \) is the orthogonal projection on the orthogonal complement of H.

This suggests picturing the unitary dilation as follows:

\[
\begin{array}{ccc}
S_1^+ & \subseteq & L^2(R_+; H_1) \\
\downarrow & & \downarrow \\
A_1 & \Rightarrow & L^2(R_+; H_2) \\
\uparrow & & \uparrow \\
H & \Rightarrow & H
\end{array}
\]

The operators \( A_1^* \) and \( B_1 \) "couple" \( (T_0, H) \) to \( (S_1^+, L^2(R_+; H_1)) \) and \( (S_1^+, L^2(R_+; H_2)) \) where \( S_1^+ \) and \( S_1^- \) are right shifts. For example \( B_1 \) is given by

\[
B_1\xi(s) = \begin{cases} 
-C + T_0 - \xi & \text{if } s \leq [0, t] \\
0 & \text{if } s \notin [0, t]
\end{cases}, \quad \xi \in D(A)
\]

It can be shown that \( B_1 \) is a contraction. In the self-adjoint case there is a simplification and it is enough to couple H to \( L^2(R; H) \) and we try to give an intuitive justification of (3.2.8).

In a physical setting the shifts will correspond to the random behaviour of the heat bath and will be the flow of Brownian motion. We expect the coupling between the system and the heat bath to be instantaneous and this coupling will take place via the coupling operator \( (2A)^{1/2} \). For \( t \geq 0 \), we therefore expect that a vector \( \begin{pmatrix} \xi(t) \\ \xi(t) \end{pmatrix} \in D(A) \) to be transformed into \( \begin{pmatrix} A \xi(t) \\ db_1 \otimes (-2A)^{1/2} \xi(t) \end{pmatrix} \) in time dt, where \( b_1 \) denotes standard Brownian motion and \( db_1 \otimes (-2A)^{1/2} \xi(t) \) is an element of \( L^2(R_+; H) \) equals \( L^2(R_+; H) \) tensor product. The second component \( db_1 \otimes (-2A)^{1/2} \xi(t) \) in integrated form is essentially \( B_1 \) in (3.2.8).

Finally, we can explain the form of \( K_s, K_R \) and \( K_e \) on physical grounds. Since the time evolution on H is self-adjoint, it does not contain a unitary part and we expect \( H_s \) to be zero. The fact that \( K_e \) should be of the form (3.2.10) follows from the same argument given above.

REFERENCES

On the Theory of Nonlinearity

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Abstract: A review of some of the recent theory is presented. A central theme is the use of state feedback and its role in a system. The relations between variediable invariance and a general decomposition p

1. Introduction
One of the most important new developments the last decade has been, without any doubt and controlled invariant distributions, a geometric approach, as it is often elegant and effective approach for solving these problems. In this regard, we mention the Disturbance Decoupling Problem. Moreover, this approach is one of the best developed theoretical picture for this problem, and therefore is valuable in these theoretic concepts like observability, and controllability.

Invariant and controlled invariant disturbances play the same role as - and in fact generally...