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NON-LINEAR FILTERING AND STOCHASTIC MECHANICS (1)

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M. Hazewinkel and J. C. Willems (eds.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, 479–503.

1. INTRODUCTION

Many of the basic ideas of non-linear filtering for diffusion processes were developed in the early and mid sixties by Stratanovich, Kushner, Wonham and others [for references, see the paper by Davis and Marcus, this volume]. Indeed this line of development could be considered as an extension of linear filtering as viewed by Kalman and Stratanovich in the sense that non-linear filtering is considered as a theory of conditional Markov processes. In the late sixties, the innovations approach to non-linear filtering was emphasized by Kailath (and Frost in his Stanford doctoral dissertation). The idea of using the innovations in an essential way in the Gaussian case dates back to Wold and Kolmogoroff in a discrete-time situation and to Bode-Shannon in the continuous situation. In this approach the observation process is first whitened in a causal and causally-invertible fashion and the form of the filter then becomes transparent. The contribution of Kailath was to see that this approach extended to a non-Gaussian situation and to conjecture that the whitening of the observation process in a causal and causally invertible manner could also be carried out in the non-linear case.

This conjecture has recently been proved by D. Allinger and the author [1] and leads to the most transparent derivation of the non-linear filtering equations (at least, when the signal and noise are independent).

It was realized by Fujisaki and Kunita that something weaker than observation-innovations equivalence would suffice to prove the basic representation results of non-linear filtering, namely that all square-integrable martingales adapted to the observations could be represented as a stochastic integral on the innovations, the integrand being square integrable and adapted to the observations. The celebrated paper of Fujisaki, Kallianpur and Kunita [2] brings to a culmination the innovations approach to non-linear filtering of diffusion processes.

This approach emphasizes the semi-martingale representation of the filter and is rooted in the intrinsic form of the Ito-differential rule due to Kunita and Watanabe. It has the disadvantage that it uses the innovations process which is an "invariant" but derived object. In most non-linear situations of interest this is not explicitly computable.

An alternative approach to the non-linear filtering problem can be traced back to the pioneering doctoral dissertations of Mortensen [3] and Duncan [4] and the important paper of Zakai [5], and is striking in its similarities to the path-space approach to Quantum Mechanics due to Feynman and its rigorization using

Wiener space ideas by Kac and Ray. In this development one works directly with the observations process and computes an unnormalized conditional density using path space integration. The computation of the actual estimate requires a further integration. The recent developments of non-linear filtering have focussed on this approach. In a previous paper [6] the author has given a systematic expository account of this point of view and many of the ideas presented in that paper has been further developed [see, for example the papers of Davis, and of Marcus and Hazewinkel in this volume]. It is also this view that led Beneš [7] to discover finite-dimensional filters for a class of non-linear filtering problems.

This viewpoint is completely consistent if not identical to the viewpoint of quantum mechanics as stochastic mechanics and quantum field theory as euchidean (stochastic) field theory. For a lucid account of these ideas see F. Guerra [8]. To make the identification, it is necessary to admit open quantum systems, that is, admit both self interactions and external interactions. Indeed, it is correct to think of the observation of a stochastic process as producing an external interaction. Just as non-trivial Markov (euclidean) fields are constructed from the free field using a Multiplicative functional transformation (see for example, Nelson [9]), similarly non-trivial filtering problems arise out of a time-dependent multiplicative functional transformation (see the article of Davis, this volume). In this framework the Kalman Filter occupies the same role as the harmonic oscillator does in quantum mechanics or the free field does in quantum field theory.

It is therefore hardly surprising that the Heisenberg algebra, the Oscillator algebra and other Lie algebras and their infinite-dimensional representations have a central role to play in this theory. Indeed in field theory, the Nelson-Feynman-Kac formula provides such a representation. The infinite-dimensional representations one seeks in filtering theory are however semi-group representations which are positivity prpeserving (leaves a certain cone invariant), and these are obtained from the Bayes formula due to Kallianpur and Striebel:

The final topic in this line of thought is the question of variational principles for non-linear filtering and the duality between filtering and control. These ideas date back to Bryson-Frazier [10] and Mortensen [11] and more recently to Hijab [12] but has never been satifactorily resolved. We show in this paper that indeed there exists a variational principle for non-linear filtering and that the equation for the unnormalized conditional density (Duncan-Mortensen-Zakai equation) is closely connected to a Bellmann-Hamilton-Jacobi equation can be replaced by a deterministic Hamilton-Jacobi equation and this is the fundamental idea behind the duality between filtering and control in the Gaussian case. These questions were touched upon in the author's previous paper [loc. cit.] but receive for the first time a complete resolution in this paper. As a by-product we obtain a stochastic variational principle for the Guerra-Nelson Stochastic Mechanics (at least in the ground state) and also stochastic control analogues of Benes' filtering problems. These ideas also allow us to study the behaviour of the unnormalized conditional density in the presence of small process and observation noise, and it is clear that there is an analogue of quasi-classical approximations of quantum mechanics in filtering theory.

The most important avenue of generalization of these ideas is in the context of diffusion processes on manifolds. This generalization seems to be necessary both for treating new filtering problems as well as stochastic mechanics.

2. FORMULATION OF NON-LINEAR FILTERING PROBLEM

The filtering problems we consider are consistent with the general model considered by Davis and Marcus in this volume, excepting we specialize the model for the signal.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with an increasing family of σ-fields. We shall generally be considering stochastic processes on a fixed time interval $[0,T]$, except in the section dealing with stochastic mechanics.

We consider the following stochastic differential system describing the model of the signal and observation processes:

$$dy_t = z_t dt + d\eta_t \qquad \text{(Observation)} \qquad (2.1)$$

$$z_t = h(x_t) \qquad \text{(Signal)} \qquad (2.2)$$

$$dx_t = b(x_t)dt + dw_t \qquad (2.3)$$

We make the following assumptions (for simplicity)

(H.1) y_t and x_t are real-valued processes

(H.2) $(w_t, \mathcal{F}_t)$ and $(\eta_t, \mathcal{F}_t)$ are independent standard Brownian motions

(H.3) $E\int_0^T |h(x_t)|^2 dt < \infty$

(H.4) x_t and η_t are independent

(H.5) $b(x_t) = f_x(x_t)$, where f_x denotes the derivative with respect to x and equation (2.3) has a unique strong solution. Further the process x is assumed to have a density p(t,x).

Remark: For much of what we do in the sequal, it is enough to assume that (2.3) has a unique weak solution and the assumption $b = f_x$ can be taken in the sense of distributions.

For simplicity we have assumed that the processes involved are scalar-valued. There is no difficulty in generalizing what follows to vector-valued processes.

Then from Theorem 6 and Example 4 of Davis and Marcus, this volume, the unnormalized conditional density $q(t,x,\omega,y_0^t)$ (where the arguments ω and y_0^t will be omitted) satisfies the stochastic partial differential equation:

$$dq(t,x) = L_0^* q(t,x)dt + L_1 q(t,x) \cdot dy_t \ , \tag{2.4}$$

where

$$\begin{cases} (L_0^*\phi)(x) = \frac{1}{2}\frac{\partial^2\phi}{\partial x^2} - \frac{\partial}{\partial x}(b(x)\phi(x)) - \frac{1}{2}h^2(x)\phi(x) \\ (L_1\phi)(x) = h(x)\phi(x) \end{cases} \tag{2.5}$$

L_0^* used later

and $\cdot$ denotes Stratonovich differential.

We assume

$$q(0,x) = q_0(x) > 0. \tag{2.6}$$

Understanding the invariance properties of equation (2.4) and its explicit solution is the fundamental problem of non-linear filtering. We however mention that computing an estimate $E[\phi(x_t) | \mathcal{F}_t^y] \triangleq \hat{\phi}_t$, where $E\int_0^T \phi(x_t)^2 dt < \infty$ requires a further integration

$$\hat{\phi}_t = \int_{\mathbb{R}} \phi(x) q(t,x)\, dx \tag{2.7}$$

and a normalization.

The remainder of the paper is devoted to an understanding of equation (2.4).

3. THE FEYNMAN-KAC FORMULA, GIRSANOV FORMULA AND GUERRA-NELSON STOCHASTIC MECHANICS

We first try to understand the <u>autonomous</u> (no external inputs) system:

$$\frac{\partial\rho(t,x)}{\partial t} = L_0^*\rho(t,x) \tag{3.1}$$

Let $\psi(x) = e^{\int_0^x b(z)dz}$ and write $\rho(t,x) = \psi(x)\hat{\rho}(t,x)$. Then $\hat{\rho}$ satisfies the equation

$$\frac{\partial\hat{\rho}}{\partial t} = \left(\frac{1}{2}\ \frac{\partial^2}{\partial x^2} - V(x)\right)\hat{\rho}(t,x), \quad \text{where} \tag{3.2}$$

$$V(x) = \frac{1}{2}h^2 + \frac{1}{2}[b_x(x) + b^2(x)] = \frac{1}{2}[f_{xx}(x) + f_x^2(x)] + \frac{1}{2}h^2(x) \tag{3.3}$$

(since b is assumed to be a gradient vector field f_x).

We remark that the operator $H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$ is a Schrodinger operator and equation (3.2) has an important role to play in Euclidean (Quantum) mechanics.

Let us make the assumption:

(H6) $V \in L^2_{loc}(\mathbb{R})$, positive, and $\lim_{|x|\to\infty} V(x) = \infty$

<u>Theorem 3.1</u>: [13, Theorem XIII.47, p.207]

The operator $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$ considered as an operator on $L^2(\mathbb{R}, dx)$ has an eigenvalue as the lowest point in its spectrum and the corresponding eigenfunction $\psi_0(x)$ is strictly positive.

The lowest eigenfunction is referred to as the ground state. Let us normalize the lowest eigenvalue to be 0 and hence the corresponding eigenfunction $\psi_0(x) > 0$ satisfies

$$-\frac{1}{2}\frac{d^2\psi_0}{dx^2} + V\psi_0 = 0$$

Then by the Feyman-Kac formula [14, Theorem x.68, p. 279]

$$\psi_0(x) = \int_W \psi_0(x_t)\exp(-\int_0^t V(x_s)ds)d\mu_w^x \qquad (3.3)$$

where $W = C(\mathbb{R}_+;\mathbb{R})$ equipped with its family of Borel sets $\mathcal{B}$ corresponding to the topology of uniform convergence on compacts and μ_w^x is Wiener measure starting at x.

Our objective is to construct a stochastic process associated with the operator $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$. Let us normalize ψ_0 such that $\int_{\mathbb{R}}|\psi_0(x)|^2dx = 1$. Define the probability measure $d\mu = |\psi_0(x)|^2dx$ and consider the space $L^2(R;\mu)$. Now, the spaces $L^2(\mathbb{R};dx)$ and $L^2(\mu)$ are unitarily equivalent under the unitary operator $U: f \to \psi_0^{-1}f: L^2(\mathbb{R};dx) \to L^2(\mathbb{R};\mu)$. On $L^2(\mathbb{R};d\mu)$ the operator H is equivalent to $H' = UHU^{-1}$ (assuming the lowest eigenvalue = 0) and $H'\phi = -\frac{1}{2}\frac{d^2\phi}{dx^2} + f_x\frac{d}{dx}$, where $f = -\ln\psi_0$.

H' is a contraction semigroup on $L^2(\mathbb{R};d\mu)$.

We now construct the stochastic differential equation defining the Markov process corresponding to the operator H' by exploiting the relationship between the Feynman-Kac Formula and the Girsanov formula and the generalized Ito-Differential rule due to Krylov and others (for the generalized Ito differential rule, see [15]).

Now the function $\psi_0 \in \mathcal{D}(H)$, we get since $f = -\ell n\psi_0$, that (i) f is continuous. (ii) f_x in the sense of distributions belongs to $L^2_{loc}(\mathbb{R};dx)$ and (iii) f_{xx} in the sense of distributions belongs to $L^1_{loc}(\mathbb{R};dx)$.

By direct calculation using $H' = UHU^{-1}$ and $f = -\ln\psi_0$, we get

$$-\frac{1}{2} f_{xx} + \frac{1}{2}(f_x)^2 = V(x) \text{ almost all } x \in \mathbb{R} \tag{3.4}$$

Now

$$\exp(-\int_0^t V(x_s)ds) = \exp(\frac{1}{2}\int_0^t f_{xx}(x_s)ds - \frac{1}{2}\int_0^t f_x^2(x_s)ds \tag{3.5}$$

Applying the generalized Ito Differential Rule to $f(x_s)$, we get

$$df(x_s) = f_x(x_s)dx_s + \frac{1}{2} f_{xx}(x_s)ds \quad . \tag{3.6}$$

Hence from (3.5) and (3.6)

$$\exp(-\int_0^t V(x_s)ds) = \exp\ [f(x_t)-f(x_0) - \int_0^t f_x(x_s)dx_s - \frac{1}{2}\int_0^t f_x^2(x_s)ds]$$

and hence

$$\exp[-\int_0^t f_x(x_s)dx_s - \frac{1}{2}\int_0^t f_x^2(x_s)dx]$$

$$= \exp(-f(x_t))\exp(f(x_0))\exp(-\int_0^t V(x_s)ds)$$

$$= \psi_0(x_0)^{-1}\psi_0(x_t)\exp(-\int_0^t V(x_s)ds) \quad .$$

Define

$$L_t = \psi_0(x_0)^{-1}\psi_0(x_t)\exp(-\int_0^t V(x_s)ds) . \tag{3.7}$$

Now L_t is (i) a well defined random variable μ_w^x - a.s. for all $x \in \mathbb{R}$ (ii) positive and from (3.3) (iii) $\int L_t d\mu_w^x = 1$.

Therefore from the properties of Wiener Process (as a Markov process), L_t is a $(W, \mathcal{B}_t, \mu_w^x)$ - martingale, where $\mathcal{B}_t$ is the sigma field generated by the coordinate functions of x after time t: Hence we may define a probability measure on $(\Omega,\mathcal{B})$ by

$$\left.\frac{d\nu^x}{d\mu_w^x}\right|_{F_t} = L_t \ .$$

Therefore by the Girsanov theorem the process w_t, $t \geq 0$ defined by

$$w_t = x_t - x_0 + \int_0^t f_x(x_s)ds$$

is a $(\mathcal{B}_t, \nu^x)$ - Brownian motion and therefore the process x_t, $t \geq 0$, considered as a stochastic process on $(\mathcal{W}, \mathcal{B}_t, \nu^x)$ is a weak solution of the stochastic differential equation

$$dx_t = -f_x(x_t)dt + dw_t \quad . \tag{3.8}$$

By construction, this equation has the unique invariant measure μ.

From our constructions, we see that

(a) $\int_{\mathcal{W}} \phi(x_t)d\nu^x = \int_{\mathcal{W}} \phi(x_t)L_t d\mu_w^x$, for all bounded continuous functions ϕ.

(b) the process x_t has a transition density $q(t,y;0,x)$ given by

$$q(t,y;0,x) = \psi_0^{-1}(x)\psi_0(y)E_{\mu_w^x}[\exp(-\int_0^t V(x_s)ds)|x_t = y] \times \times p(t,y;0,x) \quad . \tag{3.9}$$

where $E_{\mu^x}[\cdot|\cdot]$ denotes conditional expection with respect to the measure μ_w^x conditioned on $\{x_t = y\}$ and $p(t,y;0,x)$ is the transition density of the Wiener process. It follows from the work of Carmona [16] that q is a continuous functionof x and y.

We can summarize what we have done in:

<u>Theorem 3.2</u>: Under hypothesis (H6), there exists a unique family of probability measures $(\nu_x | x \in \mathbb{R})$ on the canonical probability space $(\mathcal{W},\mathcal{B})$ such that

(i) $(\mathcal{W},\mathcal{B},\mathcal{B}_t,x_t,\nu_x)$ is a strong Markov process;

(ii) the martingale problem corresponding to (3.8) has a unique solution;

(iii) the Markov process x_t has a unique invariant measure μ;

(iv) the process x_t is symmetric (i.e., the reverse Markov process is itself).

3.1 The case where V(x) is quadratic

In (2.2) and (2.3) let us assume

$$h(x) = x \tag{3.10}$$

$$b_x + b^2 = Q(x), \text{ where } Q(x) \text{ is a positive quadratic function} \tag{3.11}$$

Normalization $\frac{1}{2}(x^2-1)$

In this case V(x) = a positive quadratic function. To simplify matters, let us assume that $V(x) = x^2$. This case corresponds to the (euclidean) Harmonic oscillator. The ground state $\psi_0(x)$ can be explicitly computed to be $(\pi)^{-1/4} \exp\left(-\frac{x^2}{2}\right)$ and $H' = -\frac{1}{2}\frac{d^2}{dx^2} + x\frac{d}{dx}$ and is a contraction on $L^2(\mathbb{R}; \pi^{-1/2}\exp(-x^2)dx)$.

We note that the corresponding Markov process is the Ornstein-Uhlenbeck process

$$dx_t = -x_t dt + dw_t \tag{3.12}$$

The ideas presented in this example have played an essential role in the original discovery by Benes of explicit finite-dimensional filters for a class of non-linear problems.

Furthermore the Ornstein-Uhlenbeck operator is an example of a self-adjoint operator which generates a hypercontractive semigroup on $L^2(\mathbb{R}; d\mu)$ [14, Theorem X.56, p.260].

3.2 Discussion

Our previous development is at the heart of Nelson's stochastic mechanics. Let us first note that the operator $H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$ corresponds to the generator of a multiplicative functional of the Wiener process. What we have shown in Section 3 is that this operator is unitarily equivalent to a self-adjoint Markov semigroup on a suitable $L^2(\mathbb{R}; \mu)$ space and we have explicitly constructed that Markov process. This Markov process is symmetric in the sense that the reversed Markov process is itself. This follows from the fact that the operator unitarily equivalent to

H on $L^2(\mathbb{R};\mu)$ is self-adjoint (μ is the unique invariant measure of the Markov process). It is in this sense that the reversibility of quantum mechanics is preserved in stochastic mechanics. Moreover the expectation values of all quantum mechanical observables in the ground state that can be computed using the quantum mechanical formalism can also be computed in terms of the measure on the path space of the corresponding stochastic process. Finally, the field operators can be constructed from the stochastic formalism. We emphasize that what we have really done is shown the relationship between the Feynman-Kac formula and the Girsanov formula.

3.3 An Associated Stochastic Control Problem

Our main objective in this section is to give a variational interpretation of Nelson's stochastic mechanics. The key to this is the remark that equation (3.4) is a stationary Bellman equation arising out of a stochastic control problem.

We first consider the non-stationary situation. In equation (3.1) let us make the transformation

$$\hat{\rho}(t,x) = \exp\left(-S(t,x)\right) \tag{3.13}$$

Then $S(t,x)$ satisfies the Bellman equation:

$$\begin{aligned} \frac{\partial S}{\partial t} &= \frac{1}{2}\frac{\partial^2 S}{\partial x^2} - \frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + V(x);\ S(0,x) = S_0(x) \\ &= -\ell n\ \rho_0(x) \end{aligned} \tag{3.14}$$

It is also worth observing that $\frac{\partial S}{\partial x}$ satisfies the "Navier-Stokes-like" equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial S}{\partial x}\right) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial S}{\partial x}\right) - \dfrac{1}{2}\dfrac{\partial S}{\partial x}\dfrac{\partial}{\partial x}\left(\dfrac{\partial S}{\partial x}\right) + \dfrac{\partial V}{\partial x} \\ \dfrac{\partial S}{\partial x}(0,x) = -\dfrac{1}{\rho_0(x)}\dfrac{\partial \rho_0}{\partial x} \end{cases} \tag{3.15}$$

We now make the assumption that the potential V in addition to satisfying the previous hypotheses is convex and of class C^1 (the case of most interest is where V is an even positive polynomial).

The development that follows is essentially due to Fleming [17] and Karatzas [18] and we explicitly follow Karatzas.

Consider the stochastic control problem:

$$\begin{cases} dx_t = u_t dt + dw_t \\ x_s = x_0 \end{cases} \qquad s \le t \le T \tag{3.16}$$

As before, let $\mathcal{W} = C(0,T)$ equipped with its Borel σ-algebra and let $\mathcal{B}_t$ denote the σ-algebra generated by the coordinate functions. Consider also the σ-field $\mathcal{D}$ of subsets of $[0,T] \times \mathcal{W}$ with the property that each t-section belongs to $\mathcal{B}_t$ and each x-section is Lebesgue measurable.

An admissible control function

$$\bar{u}: ([0,T] \times \mathcal{W}, \mathcal{D}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is a measurable map such that $\bar{u}(x,t) = u_t$ and the stochastic differential equation

$$dx_t = \underline{\bar{u}(x,t)}^{t,x} dt + dw_t \tag{3.17}$$

has a unique weak solution $(x_t, w_t | 0 \le t \le T)$ on some probability space $(\Omega, F_T, P_x^u, F_t)$ for any $x \in \mathbb{R}$, with (w_t, F_t) a Wiener process and

$$E_x^u \int_0^T |\bar{u}(x,t)|^p dt < \infty$$

$$\sup_{0 \le t \le T} E_x^u |x_t|^p < \infty \quad .$$

Let U denote this class of controls.

Consider the problem of minimizing

$$J(x,t;\bar{u}) = E_x^{\bar{u}} \int_t^T \left(\frac{1}{2} u_s^2 + V(x_s)\right) dS + E_x^{\bar{u}}(S_0(x_T))$$

$$0 \le t \le T$$

(To be consistent with equation (3.14) we should reverse time; S_0 is as defined in (3.14)).

Let

$$S(x,t) = \underset{\bar{u} \in U}{\operatorname{Inf}} J(x,t,\bar{u})$$

Then S satisfies the Bellman equation

$$\begin{cases} \dfrac{\partial S}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 S}{\partial x^2} + \underset{u \in \mathbb{R}}{\min} \left(u \dfrac{\partial S}{\partial x} + \dfrac{1}{2} u^2 \right) + V(x) , \quad (x,t) \in \mathbb{R} \times [0,T] \\ S(x,T) = S_0(x) \quad . \end{cases} \tag{3.18}$$

It is shown by Karatzas [loc. cit.] that the optimal control law

$$u^*(x) = - \frac{\partial S}{\partial x} (x_t^*, t)$$

is admissible and the corresponding stochastic differential equation (3.17) has a unique strong solution.

Consider now the infinite-time problem with the cost function:

$$J(x;\bar{u}) = \lim_{T \to \infty} \frac{1}{T} E_x^{\bar{u}} \int_0^T [\frac{1}{2} u_t^2 + V(x_t^{\bar{u}})]dt \tag{3.19}$$

The minimization is now carried out over all admissible control laws which are Markovian and which given rise to an ergodic x_t process. This class is characterized by control laws $u_t = \bar{u}(x_t)$, such that $F^u(\infty) < \infty$ where

$$F^u(x) \triangleq \int_{-\infty}^{\infty} \exp\{2 \int_0^y \bar{u}(z)dz\}dy, \quad \text{and} \tag{3.20}$$

$$\int_{-\infty}^{\infty} \{\bar{u}^2(x) + V(x)\}dF^u(x) < \infty \tag{3.21}$$

Again it is shown by Karatzas that an optimal control law u^* exists such that the limit in (3.19) is independent of the starting point x. Moreover, the optimal control is given by

$$u^*(x_t^*) = - \frac{\partial S}{\partial x} (x_t^*) , \quad \text{where } S \tag{3.22}$$

is the solution of the stationary Hamilton-Jacobi equation

$$\frac{1}{2} \frac{\partial^2 S}{\partial x^2} - \frac{1}{2} \left(\frac{\partial S}{\partial x}\right)^2 + V(x) = 0 \quad . \tag{3.23}$$

It is interesting to note that from our previous considerations, (see equation (3.[illegible]) and the definition of f) we know the optimal control explicitly,

$$u^*(x) = (\psi_0(x))^{-1} \frac{\partial \psi_0}{\partial x}, \tag{3.24}$$

where ψ_0 is the non-degenerate ground state of the operator $-\frac{1}{2}\frac{d^2}{dx^2} + V(x)$.

In the case $V(x) = \frac{1}{2}x^2$, we get $u^*(x) = -x$, a well known result in stochastic control theory.

Our final observation is that to this class of finite-time stochastic control problems we can rigorously apply Bismut's duality theory of stochastic control [19]. According to this theory, there exists an adapted right continuous process p_t and and an adapted measurable process π_t, $E\int_0^T \pi_t^2 dt < \infty$ such that

$$\begin{cases} dx_t = u_t^* dt + dw_t \\ x_0 = x \end{cases} \tag{3.25}$$

$$\begin{cases} dp_t = -\frac{\partial V}{\partial x}(x_t)dt + \pi_t dw_t \\ p_T = \frac{\partial S_0}{\partial x}(x_T) \end{cases} \tag{3.26}$$

$$u^* = \arg\min_{u \in \mathbb{R}} \left(\frac{1}{2}u^2 + u.p\right) \tag{3.27}$$

Equations (3.25) - (3.27) are the "stochastic" bi-characteristics of the Bellman equation.

What we have shown in this section is that certain multiplicative functionals of Brownian motion (and indeed more general Markov processes) have associated stochastic canonical equations of motion, and in this sense stochastic mechanics is exactly like classical mechanics.

4. A NON-LINEAR STOCHASTIC CONTROL PROBLEM WITH AN EXPLICIT SOLUTION

We now consider a class of stochastic control problems which are the analogues of non-linear filtering problems first considered by Benes which have an explicit solution. This solution is obtained by exploiting the ideas of Section 3. A prototype example of this class is:

$$dx_t = f(x_t)dt + u_t dt + dw_t \tag{4.1}$$

where f satisfies the Riccati equation

$$\frac{df}{dx} + f^2 = x^2 \quad . \tag{4.2}$$

The cost function is

$$J(t,x;u) = E_x^u\left(\frac{1}{2}x_T^2\right) + E_x^u\int_t^T\left(\frac{1}{2}u_s^2 + \frac{1}{2}x_s^2\right)ds$$

Check !

We shall place ourselves under the hypotheses of the previous section for the control laws.

If $S(t,x) = \underset{u}{\text{Inf.}}\ J(t,x;u)$, then the Bellman equation for S is

$$\begin{cases} \dfrac{\partial S}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 S}{\partial x^2} + f\dfrac{\partial S}{\partial x} - \dfrac{1}{2}\left(\dfrac{\partial S}{\partial x}\right)^2 + \dfrac{1}{2}x^2 \\ S(T,x) = \dfrac{1}{2}x^2 \end{cases} \tag{4.3}$$

Let us introduce the transformation

$$S(t,x) = -\ln \rho(t,x)$$

Then ρ satisfies the equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \rho}{\partial x^2} + f\dfrac{\partial \rho}{\partial x} - \dfrac{1}{2}x^2\rho \\ \rho(T,x) = \exp\left(-\dfrac{x^2}{2}\right). \end{cases} \tag{4.4}$$

Now we remove the drift term in the above equation by introducing the Gauge transformation

$\rho(t,x) = \psi(x)\hat{\rho}(t,x)$, where

$\psi \in C^{\infty}(\mathbb{R})$ invertible is to be chosen. A direct computation shows that $\hat{\rho}$ satisfies:

$$\frac{\partial\hat{\rho}}{\partial t} = \frac{1}{2}\frac{\partial^2\hat{\rho}}{\partial x^2} + \left(\psi^{-1}\frac{\partial\psi}{\partial x} + f\right)\frac{\partial\hat{\rho}}{\partial x} + \left(\frac{1}{2}\psi^{-1}\frac{\partial^2\psi}{\partial x^2} + \psi^{-1}\frac{\partial\psi}{\partial x}f - \frac{1}{2}x^2\right)\hat{\rho} \tag{4.5}$$

$$\hat{\rho}(T,x) = \psi^{-1}\rho(T,x) \quad ,$$

Now choose ψ to satisfy

$$\frac{\partial\psi}{\partial x} + f\psi = 0$$

Then $\hat{\rho}$ satisfies

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} &= \frac{1}{2}\frac{\partial^2\hat{\rho}}{\partial x^2} - \frac{1}{2}\left(\frac{\partial f}{\partial x} + f^2 + x^2\right)\hat{\rho} \\ &= \frac{1}{2}\frac{\partial^2\hat{\rho}}{\partial x^2} - \frac{1}{2}x^2\hat{\rho}\end{aligned} \tag{4.6}$$

This backward equation has a unique solution given by the Feynman-Kac formula:

$$\hat{\rho}(t,x) = E_{tx}[\hat{\rho}(x_T)\exp(-\int_t^T \frac{1}{2}x_s^2 ds)], \text{ where} \tag{4.7}$$

E_{tx} denotes expectation with respect to Wiener measure conditioned on $x_t = x$.

The integration in (4.7) can be carried out by using Gaussian integrals or by using the method of bicharacteristics introduced by the author in [6] (see section 3.2). By transforming back we get an explicit solution for S.

The developments in this section show that the fundamental solution to (4.2) can be written down in terms of a Riccati equation that arises in the gain computation in optimal control and Kalman filtering (not to be confused with the Riccati equation (4.2)).

5. LIE ALGEBRAIC CONSIDERATIONS

The fundamental Hamiltonian in the previous considerations is the Hamiltonian $H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$ acting on $L^2(\mathbb{R}; dx)$ or the unitarily equivalent operator $H = -\frac{1}{2}\frac{d^2}{dx^2} + x\frac{d}{dx}$ acting on $L^2(\mathbb{R}; dg)$ where g is Gauss measure. This Hamiltonian corresponds to the Harmonic Oscillator. Underlying the Harmonic oscillator is the solvable Lie algebra with basis $\{r,p,q,i\}$ the oscillator algebra whose commutation relations are

$$\begin{cases} [r, p] = q \\ [p, q] = i \\ [r, q] = p . \end{cases} \tag{5.1}$$

A representation of this Lie Algebra by a Lie Algebra of unbounded operators is obtained by the correspondence

$$r \to L_0 = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$$

$$q \to L_1 = x$$

$$p \to L_2 = \frac{d}{dx}$$

$$i \to L_3 = I$$

Let T denote this representation. Let G denote the connected Lie group whose Lie algebra is the oscillator algebra. We are interested in a representation π of G by bounded operators on $L^2(\mathbb{R}; g)$, such that $\pi(g)$, $g \in G$ is a C_0-semi-group and $\pi(e^{tr}) = e^{-tT(r)}$ is a positivity preserving contraction semi-group. The semi-group e^{-tL_0} can be constructed via the Feynman-Kac formula. The interest of the Lie algebraic viewpoint is that a finite dimensional sufficient statistic can be obtained for evaluating

$(e^{-tL_0}\rho_0)(x)$, by considering the basis

$\{\frac{1}{2}\frac{d^2}{dx^2}, \frac{1}{2}x^2, x\frac{d}{dx}, I\}$ for the oscillator algebra and writing

in analogy with the Wei-Norman theory (as first suggested by Brockett)

$$(e^{-tL_0}\rho_0)(x) = [e^{g_1(t)\frac{d^2}{dx^2}} e^{g_2(t)x^2} e^{g_3(t)x\frac{d}{dx}} e^{g_4(t)} \rho_0](x)$$

and obtaining differential equations for g_1, g_2, g_3 and g_4. Indeed g_1 is t and required to be nonnegative. For the oscillator' algebra this has been rigorously proved in the doctoral dissertation of D. Ocone [20].

Consider now the Lie algebra of operators with generator $-\frac{1}{2}\frac{d^2}{dx^2} + V(x)$ and x , with V an even positive polynomial (but not quadratic). The work of Avez and Heslot [21] suitably modified shows that the corresponding Lie algebra is infinite-dimensional and simple. Hence it is unlikely that there are other examples of multiplicative functions of Brownian motion whose semigroups can be constructed using a finite dimensional sufficient statistic.

Indeed, it is not difficult to show that the only perturbations one can allow in the operators L_0 and L_1 so that the Lie algebra remains finite dimensional are $a\frac{d}{dx}$, x, x^2 (or their linear combinations; the only variable coefficient first order linear differential operator that we can allow must satisfy the condition $\frac{da}{dx} + a^2$ = quadratic).

6. NON-LINEAR FILTERING

It is a pleasant fact that the ideas expressed in the previous sections generalize in a very natural way to non-linear filtering theory. The main observation to make in that non-linear filtering theory corresponds to the (euchidean) quantum mechanical situations when we allow time-dependent random external interactions (in addition to self-interactions).

6.1 Pathwise Non-Linear Filtering

To give a variational interpreation of the non-linear problem it is necessary to consider the pathwise solution to non-linear filtering problem originally initiated by Clark (c.f. the paper of Davis this volume). For our purpose we could think , that in this approach, the stochastic integral in equation(2.4) is removed by a time-dependent Gauge transformation.

Define

$$\tilde{q}(t,x) = \exp(-h(x)y_t)q(t,x) \tag{6.1}$$

Then $\tilde{q}(t,x)$ satisfies

$$\begin{cases} \dfrac{d\tilde{q}(t,x)}{dt} = \exp(-h(x)y_t)\,[L_0^* - \frac{1}{2}L_1^2]\,(\exp(h(x)y_t\,\tilde{q}(t,x)) \\ \tilde{q}(0,x) = \tilde{q}_0(x) = q_0(x) > 0 \end{cases} \tag{6.2}$$

We consider equation (6.2) for a fixed $y_{(\cdot)} \in C(0,T;\mathbb{R})$ and indeed if necessary we could approximate $y_{(\cdot)}$ by smoother functions.

It is convenient to write equation (6.2) in two other equivalent forms

$$\frac{d\tilde{q}(t,x)}{dt} = (L_0^* - \frac{1}{2}L_1^2)\tilde{q}(t,x) + y_t L_2\tilde{q}(t,x) - y_t^2 L_3\tilde{q}(t,x) \tag{6.3}$$

where

$$L_2 = [\frac{1}{2}L_0^* - \frac{1}{2}L_1^2, L_1] = \frac{dh}{dx}\frac{d}{dx} + \left(\frac{1}{2}\frac{d^2h}{dx^2} - b\frac{dh}{dx}\right)$$

$$L_3 = [L_1, L_2] = -\left(\frac{dh}{dx}\right)^2$$

$$\tilde{q}(0,x) = \tilde{q}_0(x) = q_0(x) > 0 .$$

and

$$\begin{cases} \dfrac{\partial\tilde{q}}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2\tilde{q}}{\partial x^2} - \tilde{b}(t,x)\dfrac{\partial\tilde{q}}{\partial x} - V(t,x)\tilde{q}(t,x) \\ \tilde{q}(0,x) = q_0(x) > 0 , \end{cases} \tag{6.4}$$

where

$$\begin{cases} \tilde{b}(t,x) = b(x) - y_t\dfrac{dh}{dx} \\ V(t,x) = -\dfrac{1}{2}y_t^2\left(\dfrac{dh}{dx}\right)^2 + y_t b(x)\dfrac{dh}{dx} + \dfrac{db}{dx} + \dfrac{1}{2}h^2(x) . \end{cases} \tag{6.5}$$

Equation (6.3) exhibits the role of the commutators of $L_0^* - \frac{1}{2}L_1^2$ and L_1 and equation (6.4) shows that basically we are dealing with a situation not unlike that considered in Section 3.

6.2 Transformation into a Stochastic Control Problem

As in Section (3.3), introduce the transformation

$$\tilde{q}(t,x) = \exp(-S(t,x)) \tag{6.6}$$

Then $S(t,x)$ satisfies the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \dfrac{\partial S}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 S}{\partial x^2} - \tilde{b}(t,x)\dfrac{\partial S}{\partial x} - \dfrac{1}{2}\left(\dfrac{\partial S}{\partial x}\right)^2 + V(t,x) \\ S(0,x) = S_0(x) = -\ln q_0(x) \end{cases} \tag{6.7}$$

Equation (6.7) corresponds to the stochastic optimal control problem

$$\begin{cases} dx_t = u_t dt + dw_t \\ x_s = x_0 \end{cases} \qquad s \le t \le T \tag{6.8}$$

with cost function

$$J(s,x_0;\bar{u}) = E\left[\int_s^T L(t,x_t,u_t)dt + S_0(x_T)\right],$$

where

$$L(t,x,u) = \frac{1}{2}[u + \tilde{b}(t,x)]^2 + V(t,x)$$

and the minimization is to be carried out over the class of Markov feedback controls

$$u_t = \bar{u}(t,x_t).$$

satisfying the conditions in section 3.3. To be consistent we should reverse time, as remarked in Section 3.

If h is a polynomial and if the drift b satisfies some mild conditions, then using the work of Fleming [22], it is possible to show that equation (6.7) has a solution with appropriate regularity. In this way we can prove that equation (6.4) has a solution in the strong (classical) sense. To prove uniqueness of (6.4) we can invoke a maximum principle argument directly on equation (6.4) (cf. [23] for maximum principles for equations with unbounded coefficients). It is worthwhile making the remark that transforming the Bellman equation into an equivalent Zakai equation and analyzing it may also be a useful tool for stochastic control problems. The details of these ideas will be presented in a joint paper with Wendell Fleming.

The relation between the pathwise equations of non-linear filtering and stochastic control introduced in this section explains in the clearest possible way the "duality" that exists between filtering and control. Previous difficulties in defining a likelihood functional because of the non-differentiability of Wiener paths are completely avoided using the pathwise equation (6.2) for fixed $y \in C(0,T;\mathbb{R})$.

6.3 Various Examples

<u>Example 1</u> (Kalman Filtering)

$$\begin{cases} x_t = w_t \\ dy_t = x_t dt + d\eta_t \end{cases} \tag{6.10}$$

Here w_t and η_t are standard Brownian motions which are independent. Then from (6.5)

$$\tilde{b}(t,x) = -y_t$$

$$V(t,x) = -\frac{1}{2} y_t^2 + \frac{1}{2} x^2, \text{ and hence from (6.9)}$$

$$L(t,x_t,u_t) = \frac{1}{2} [u_t - y_t]^2 - \frac{1}{2} y_t^2 + \frac{1}{2} x_t^2 = \frac{1}{2} u_t^2 + \frac{1}{2} x_t^2 - u_t y_t$$

and we have a stochastic control problem with a quadratic cost criterion. The theory of this is essentially the same as the theory of deterministic linear optimal control with a quadratic cost criterion.

<u>Example 2</u> (Bilinear Filtering)

$$\begin{aligned} dx_t &= x_t\, dw_t \\ dy_t &= x_t dt + d\eta_t, \end{aligned} \tag{6.11}$$

with the same hypothesis as in Example 1.

For this problem the pathwise equation of non-linear filtering is:

$$\frac{\partial \tilde{q}}{\partial t} = \frac{1}{2} x^2 \frac{\partial^2 \tilde{q}}{\partial x^2} + (2x-y)\frac{\partial \tilde{q}}{\partial x} + \frac{1}{2} y^2 - \frac{1}{2} x^2 + 1 \tag{6.12}$$

and the corresponding Bellman equation is:

$$\frac{\partial S}{\partial t} = \frac{1}{2} x^2 \frac{\partial^2 S}{\partial x^2} + (2x-y_t) \frac{\partial S}{\partial x} - \frac{1}{2} x^2 \frac{\partial^2 S}{\partial x^2} + \frac{1}{2} x^2 - \frac{1}{2} y_t^2 - 1 \tag{6.13}$$

and the stochastic control problem is:

$$dx_t = x_t(u_t + dw_t) \tag{6.14}$$

$$L(t,x_t,u_t) = \frac{1}{2} (u_t + 2x_t - y_t)^2 + \frac{1}{2} x_t^2 - \frac{1}{2} y_t^2 - 1$$

These calculations also show that although the perfectly observable LQG-stochastic control problem with state dependent noise has a linear solution, this problem does not have a corresponding "dual" filtering problem.

Remarks

We may apply the Davis-Varaiya theory or the Bismut theory to obtain necessary conditions of optimality for problem (6.8) - (6.9) in the form of a maximum principle. This would give rise to stochastic bi-characteristics for equation (6.7) and in general these are necessary to compute the solution of (6.7) or equivalently (6.4). It appears that only for Kalman filtering or Benes problems is it possible to reduce these to the characteristics of an ordinary Hamilton-Jacobi equation parametrized by dy_t. This was done in the author's earlier paper cited in the introduction. The reason for this is the fact that the theorqy of deterministic LQ-control and perfectly observable LQG-control is essentially the same.

6.4 Remarks on Approximations and Perturbation Theory

The relationship between non-linear filtering and stochastic control appears to clarify various approximation schemes currently used and provides guidelines for their systematic analysis. For example, if in (6.7) b is approximated by a linear function in x and V is approximated by a quadratic function in x, locally in t, then we have a Kalman filtering problem for a small time interval $[0,\tau]$, say. Having obtained $S(t,x)$, $t \in [0,\tau]$ the above approximation and iteration procedure can be continued. Thus the extended Kalman Filtering algorithm is the analogue of Newton's method.

Furthermore, the study of filtering problems of the form

$$dx_t = b(x_t)dt + \sqrt{\varepsilon}\, dw_t$$

$$dy_t = h(x_t)dt + \sqrt{\varepsilon}\, d\eta_t$$

can be reduced to the study of the corresponding stochastic control problems with small parameter ε and the asymptotic behaviour of $S(t,x;\varepsilon)$ as a function of ε can be studied using methods developed by Fleming.

6.5 Lie Algebraic Considerations

It is clear from the previous development that the Lie algebraic approach to the study of non-linear filtering problems is entirely analogous to our considerations in section 5 for the autonomous system. The reason for this is that the noise enters the Zakai equation in a "finite-dimensional" way and only has the function of parametrizing the manifold in which the Zakai equation is evolving. Therefore the Lie algebrda of the Kalman filter is the oscillator algebra and the class of examples considered by Benes give rise to Lie algebras which are gauge equivalent to the oscillator algebra.

7. FINAL REMARKS

We have shown in this paper that a close relationship exists between non-linear filtering theory and stochastic Hamilton-Jacobi theory. This work requires generalization in the direction of the study of filtering and control problems on Riemannian manifolds. A beginning in this direction has already been made by Duncan ([24], [25]). The most intriguing possibility however is to discover the analogues of completely integrable Hamiltonian systems in non-filtering and stochastic control.

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Note added in proofs. The fact that the process corresponding to the optimal value function is a martingale has an important implication in this theory. Indeed a martingale is the analogue of a 'conserved quantity' and the interpretation says that the optimal conditional energy process is analogous to an integral of motion.