REFERENCES


Generic Solvability of Morgan's Problem

MICHAEL E. WARREN AND SANJOY K. MITTER

Abstract—For m-input, m-output, linear time-invariant systems of the form

\[ \dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t) \]

it is shown that Morgan's problem, (decoupling into an m-single input-single output subsystem) is solvable for almost all real matrix triples \((A, B, C)\) consistently dimensioned.

I. INTRODUCTION

We consider linear, time invariant, multivariable systems of the form

\[ \dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t) \]

(1)

where \(x(t) \in \mathbb{R}^n\), \(u(t)\), and \(y(t) \in \mathbb{R}^m\), with \(A, B,\) and \(C\) appropriately dimensioned real matrices. The problem of decoupling an m-input, m-output system (1) into m scalar input-scalar output subsystems by the use of a feedback control law

\[ u(t) = Fx(t) + Gv(t) \]

(2)

is first considered by Morgan [1] who constructed a sufficient condition for decoupling. In 1967, Falb and Wolovich [2] completely solved this question showing that for almost all parameter sets \((A, B, C)\) satisfying several loose dimensional constraints, the subspaces

\[ \mathcal{S}_i = \{x \in \mathbb{R}^n : \text{for } i \in m \} \]

(3)

will be controllability subspaces.

Relying heavily upon the work of Fabian and Wonham, we will show that (5) is generically true for systems of the form (1). That is, if we consider real matrix triples \((A, B, C)\) as points in \(\mathbb{R}^{n(n+m+m)}\), then those points at which (5) fails to hold lie on a proper algebraic variety in \(\mathbb{R}^{n(n+m+m)}\) and hence constitute a set of zero measure.

II. PROBLEM FORMULATION

For any positive integer \(k\), let \(k\) denote the set \(\{1,2,\ldots,k\}\). We designate the \(i\)th row of \(C\) from (1) by \(C_i\) for \(i \in m\). For each \(C_i\) define the nonnegative integer \(d_i\) and the row vector \(D_i\):

\[ d_i = \min \{ j : C_iA^jB = 0, \quad j = 0, 1, \ldots, n-1 \} \]

(4)

\[ d_i = n-1, \quad \text{if } C_iA^jB = 0 \quad \text{for all } j > 0 \]

(5)

\[ D_i = C_iA^{d_i}B \]

(6)

(For the discrete time analog of (1), \(d_i+1\) represents the minimum time delay for the effect of any input to be visible at output \(i\), and \(D_i\) represents the first nontrivial pointwise mapping from inputs to output \(i\).)

Morse and Wonham [3,4] developed a more general theory of decoupling and found an equivalent geometric condition for the solution of Morgan's problem. Fabian and Wonham [5] extended these geometric results to show, in particular, that (1) is generically (i.e., for almost all parameter sets \(A, B,\) and \(C\) decoupleable if dynamic compensation is allowed. Building upon the machinery of [5], we will show that Morgan's problem itself is generically solvable.

III. MAIN RESULT

Let \(N = n(n+m+m)\) and consider the ring of polynomials in \(N\) indeterminates over the reals, \(\mathbb{R}[\lambda_1, \ldots, \lambda_N]\). An algebraic variety \(V \subset \mathbb{R}^N\) is the set of common zeros of a finite number of such polynomials. A variety is called proper if it is not equal to \(\mathbb{R}^N\), and nontrivial if it is not empty.

A property \(\Pi\) is a function on \(\mathbb{R}^N\) to a two element set, \{true, false\} for example. If \(V\) is a proper variety of \(\mathbb{R}^N\), we say \(\Pi\) is generic relative to \(V\) if \(\Pi\) is true everywhere on \(R^N\) except for a subset of \(V\). \(\Pi\) is deemed generic if such a \(V\) exists. Since a proper variety is closed in the usual topology, it follows that if \(\Pi\) is generic relative to \(V\), for every \(x \in V\) (the complement of \(V\)) \(\Pi\) is true on some neighborhood of \(x\). As a proper variety \(V\) cannot contain any open set in \(R^N\) (if this were so, the defining polynomials would all be identically zero). It follows that if \(\Pi\) is false for some \(x \in \mathbb{R}^N\), then there exist points arbitrarily close to \(x\) such that \(\Pi\) is true at these points.

The key lemma of our development will be applicable to a more general class of linear systems than only \(m\) input--\(m\) output systems. Indeed consider a matrix triple \((A, B, C)\) representing a system of the form (1) with \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times n}\), and \(C \in \mathbb{R}^{n \times n}\). We assume an arbitrary partition of \(C\) into \(k\) submatrices.
(4) then (5) is generically true.

Proof. We shall make use of the results on the generic dimensions of subspaces from [5] without specific reference. Further identities which hold everywhere except possibly on a subset of a proper algebraic variety will be indicated by a postscripted (g).

We note that any \( r \times s \) matrix \( Q \) generically has rank \( r \min(s, r) \). For otherwise all \( i \times i \) minors of \( Q \) must vanish identically, in which case the elements of \( Q \) constitute a zero for a set of polynomials defined on \( R^{r \times s} \).

Thus, letting \( \mathcal{C}_i \) denote the image of the map \( C_i \), we have \( \dim \mathcal{C}_i = q_i \) (g) and \( \dim \ker C_i = n - q_i \) (g) for all \( i \in k \). Then from (6)

\[
\dim \left( \sum_{j \in i} \mathcal{C}_i \right) = \min(n, q^*_i) = q^*_i \quad \text{for all } i \in k.
\]

where

\[
q^*_i = \sum_{j \cap i} q_j
\]

and by complementation

\[
\dim \mathcal{B}_i = n - \min(n, q^*_i) = n - q^*_i \quad \text{for all } i \in k.
\]

We will first demonstrate that under the hypothesis of the lemma

\[
\sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i = \mathcal{B}_k \quad \text{for all } k \in \mathbb{R}.
\]

Since \( \sum_{i \in k} (\mathcal{B}_i \cap \mathcal{X}_i) \subset \mathcal{B}_k \), we need only prove that (6) implies

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) - \dim \mathcal{B}_k = m(g).
\]

Using the geometric identity

\[
\dim (\mathcal{B} \cap \mathcal{Y}) = \dim (\mathcal{B}) + \dim (\mathcal{Y}) - \dim (\mathcal{B} \cap \mathcal{Y})
\]

to expand the left side of (8) results in

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) = \dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) - \dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) - \dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right).
\]

But

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) - \dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) = \dim \mathcal{X}_k + \dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) - \dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right)
\]

which yields

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) = \dim \mathcal{B}_k + \dim \mathcal{X}_k - \dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right)
\]

Now

\[
\dim \left( \mathcal{B} \cap \mathcal{X}_k \right) = \dim \mathcal{B} + \dim \mathcal{X}_k - \dim \left( \mathcal{B} \cap \mathcal{X}_k \right)
\]

\[
= \min(n, m + q^*_k) = m - q^*_k \quad \text{for all } k \in \mathbb{R}
\]

as \( m > q^*_k \) by (6). Since

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) = \min(n, \dim \mathcal{B} + \dim \mathcal{X}_k) \quad \text{for all } \mathcal{B}, \mathcal{X}_k
\]

it follows from (10) that

\[
\dim \left( \mathcal{X}_k + \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) = \min \left( n, m + q^*_k \right) = m - q^*_k \quad \text{for all } \mathcal{B}, \mathcal{X}_k
\]

Thus, letting \( \ker C_i = q_i \) (g) and \( \ker C_i = n - q_i \) (g) for all \( i \in k \). Then from (6)

\[
\dim \left( \sum_{j \in i} \mathcal{C}_i \right) = \min(n, q^*_i) = q^*_i \quad \text{for all } i \in k
\]

where

\[
q^*_i = \sum_{j \cap i} q_j
\]

and by complementation

\[
\dim \mathcal{B}_i = n - \min(n, q^*_i) = n - q^*_i \quad \text{for all } i \in k.
\]

We will first demonstrate that under the hypothesis of the lemma

\[
\sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i = \mathcal{B}_k \quad \text{for all } k \in \mathbb{R}
\]

Since \( \sum_{i \in k} (\mathcal{B}_i \cap \mathcal{X}_i) \subset \mathcal{B}_k \), we need only prove that (6) implies

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) - \dim \mathcal{B}_k = m(g).
\]

Using the geometric identity

\[
\dim (\mathcal{B} \cap \mathcal{Y}) = \dim (\mathcal{B}) + \dim (\mathcal{Y}) - \dim (\mathcal{B} \cap \mathcal{Y})
\]

to expand the left side of (8) results in

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) - \dim \mathcal{B}_k = m(g).
\]

But

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) - \dim \mathcal{B}_k = m(g)
\]

which yields

\[
\dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right) = \dim \mathcal{B}_k + \dim \mathcal{X}_k - \dim \left( \sum_{i \in k} \mathcal{B}_i \cap \mathcal{X}_i \right)
\]

Now

\[
\dim (\mathcal{B} \cap \mathcal{X}_k) = \dim (\mathcal{B} + \dim \mathcal{X}_k - \dim (\mathcal{B} \cap \mathcal{X}_k)
\]

\[
= m + (n - q^*_k) - \min(n, m + n - q^*_k) \quad \text{for all } \mathcal{B}, \mathcal{X}_k
\]

\[
= m - q^*_k \quad \text{for all } \mathcal{B}, \mathcal{X}_k
\]

REFERENCES


On The Decomposition of State Space

P. E. DRENNICK

Abstract—The response of a linear control system is often viewed as a superposition of independent, modal responses. For complex systems, traditional techniques for resolving modal responses may be either inapplicable, quite expensive, or numerically unstable.

Manuscript received February 14, 1974; revised September 9, 1974. This work was supported in part by the Air Force Office of Scientific Research under Contract AFOSR-72-2389. The author is with Operations Research, International Nickel Company, New York, N. Y. 10004.