Profit Loss in Cournot Oligopolies

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Abstract

We consider a Cournot oligopoly model where multiple suppliers (oligopolists) compete by choosing quantities. We compare the aggregate profit achieved at a Cournot equilibrium to the maximum possible, which would be obtained if the suppliers were to collude. We establish a lower bound on the profit of Cournot equilibria in terms of a scalar parameter derived from the inverse demand function and the number of suppliers. We also provide another lower bound that depends on the maximum of the suppliers' market shares. The lower bounds are tight when the inverse demand function is affine. Our results provide nontrivial quantitative bounds on the loss of aggregate profit for several inverse demand functions that appear in the economics literature.

Keywords: Cournot oligopoly, revenue management, Nash equilibrium
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1. Introduction

We consider a Cournot oligopoly model where multiple suppliers (oligopolists) compete by choosing quantities. Our objective is to compare the total profit earned at a Cournot equilibrium to the maximum possible total profit, which would be obtained if the suppliers were to collude.

1.1. Background

It is well known that oligopolists can collude by jointly restricting their output and thereby increase their total profit (Chamberlin (1929); Friedman (1971)). There is a large literature on collusive behavior in oligopolistic markets. For example, Green and Porter (1984) show that in the presence of demand uncertainty, it may be possible for suppliers to form a self-policing cartel to maximize their joint profits. Also, some recent works show that forward trading may raise the prices (Mahenca and Salanie (2004)) and may allow suppliers to sustain collusive profits (Liskia and Montero (2006)).

In this paper, we focus on the classical static Cournot oligopoly model, and explore the profit loss due to competition. We compare the aggregate profit earned at a Cournot equilibrium to the maximum possible profit, that is, the aggregate profit that would have been achieved if the suppliers were to collude. Oligopolist profit loss due to competition has received some recent attention. Anderson and Renault (2003) quantify the profit loss in Cournot oligopoly models with concave demand functions. However, most of their results focus on the relation between consumer surplus, producer surplus, and the aggregate social welfare achieved at a Cournot equilibrium, rather than on the relation between the aggregate profit achieved at a Cournot equilibrium and the maximum aggregate profit. Perakis and Sun (2011) study supply chains with partial positive externalities and show that the profit loss at an equilibrium is at least 25% of the maximum profit.

Other recent works have reported various bounds on the profit loss at an equilibrium for oligopoly models with affine demand functions. For a differentiated oligopoly model, Farahat and Perakis (2009) establish lower and upper bounds on the profit loss at an equilibrium of price (Bertrand) competition. Closer to the present paper, Kluberg and Perakis (2008) compare the aggregate profit earned by the suppliers under Cournot competition to the corresponding maximum possible, for the case where suppliers produce multiple differentiated products and the demand is an affine function of the price. However, one of their key assumptions does not hold in the Cournot model studied in this paper[1]. We finally note that this work is related to the complementary literature that studies the social welfare loss at Cournot equilibria (Johari and Tsitsiklis, 2005; Corchon, 2008; Tsitsiklis and Xu, 2011).

1.2. Our contribution

In this paper, we study the profit loss in a classical Cournot oligopoly model, for a broad class of nonincreasing inverse demand functions that yield concave revenue functions. We establish a lower bound of the form $f^P(c/d, N)$ on the profit ratio of a Cournot equilibrium (the ratio of the aggregate profit earned at the equilibrium to the maximum possible). Here, f^P is a function given in closed form, c is the absolute value of the slope of

[1]In our model, the matrix $\mathbf{B}$ in the inverse demand function (a notation used in Kluberg and Perakis (2008); Kluberg (2011)) is an $N \times N$ matrix (N is the number of suppliers) with all its elements equal to 1 and is therefore not invertible (cf. Chapter 2.2 of Kluberg (2011)).

the line that agrees with the inverse demand function at a profit-maximizing output and at the Cournot equilibrium, d is the absolute value of the slope of the inverse demand function at the Cournot equilibrium, and N is the number of suppliers. We also derive another form of profit ratio lower bounds, $g^P(c/d, r)$, which does not depend on the number of suppliers, but on the market share of the largest supplier at the equilibrium, r.

For Cournot oligopolies with affine inverse demand functions, we have $c/d = 1$, and our lower bounds are tight. More generally, the ratio c/d can be viewed as a measure of nonlinearity of the inverse demand function. As the parameter c/d goes to infinity, the lower bounds converge to zero and arbitrarily high profit losses are possible. Our results allow us to lower bound the profit ratio of Cournot equilibria for a large class of Cournot oligopoly models in terms of qualitative properties of the inverse demand function, without having to restrict to the special case of affine demand functions, and without having to calculate the equilibrium and the profit-maximizing output. Furthermore, our results could be useful for monitoring, detecting, or penalizing collusion (cf. page 130 of Phlips (1995)). For instance, if our bounds indicate that the Cournot equilibrium profit is already close to the maximum possible, collusion is not a concern. On the contrary, in the opposite case there would be good reason for close monitoring and stronger penalties for collusion.

The general methodology used in this paper is similar in spirit to that used in a companion paper (Tsitsiklis and Xu, 2011) to derive lower bounds on the efficiency (upper bounds on the welfare loss) of Cournot equilibria. Furthermore, the development runs along similar lines. However, the assumptions, the details, and the expressions in the various results are different. For instance, the assumption in Tsitsiklis and Xu (2011) that the inverse demand function is convex is replaced here by an assumption that a monopolist's revenue is a concave function of the price (Assumption 4).

1.3. Outline of the paper

The rest of the paper is organized as follows. In the next section, we formulate the model. In Appendix A, we provide some mathematical preliminaries on Cournot equilibria that will be useful later, including the fact that profit ratio lower bounds can be obtained by restricting to linear cost functions. We also show that for the purpose of studying the worst case profit loss, it suffices to restrict to a special class of piecewise linear inverse demand functions. This leads to our main results, lower bounds on the profit ratio of Cournot equilibria (Theorems 1 and 2 in Section 3). Based on these theorems, in Section 4 we derive a number of corollaries that provide profit ratio lower bounds that can be calculated without detailed information on these equilibria. We apply these results to various commonly encountered inverse demand functions. Finally, in Section 5, we make some brief concluding remarks.

2. Formulation

In this section, we first define the Cournot oligopoly model that we study in this paper, and then introduce several main assumptions that we will be working with, and some definitions.

We consider a market for a single homogeneous good with inverse demand function $p : [0, \infty) \to [0, \infty)$ and N suppliers. Supplier $n \in \{1, 2, \dots, N\}$ has a cost function $C_n : [0, \infty) \to [0, \infty)$. Each supplier n chooses a nonnegative real number x_n, which is the amount of the good to be supplied by her. The **strategy profile** $\mathbf{x} = (x_1, x_2, \dots, x_N)$ results in a total supply denoted by $X = \sum_{n=1}^{N} x_n$, and a corresponding market price $p(X)$. Supplier n's payoff is

$$\pi_n(x_n, \mathbf{x}_{-n}) = x_n p(X) - C_n(x_n),$$

where we have used the standard notation $\mathbf{x}_{-n}$ to indicate the vector $\mathbf{x}$ with the component x_n omitted. A strategy profile $(x_1, x_2, \dots, x_N)$ is a Cournot (or Nash) equilibrium if

$$\pi_n(x_n, \mathbf{x}_{-n}) \geq \pi_n(x, \mathbf{x}_{-n}), \qquad \forall\, x \geq 0, \;\; \forall\, n \in \{1, 2, \dots, N\}.$$

In the sequel, we denote by f' and f'' the first and second, respectively, derivatives of a scalar function f, if they exist. For a function defined on a domain $[0, Q]$, the derivatives at the endpoints 0 and Q are defined as left and right derivatives, respectively. For points in the interior of the domain, and if the derivative is not guaranteed to exist, we use the notation $\partial_+ f$ and $\partial_- f$ to denote the right and left, respectively, derivatives of f; these are guaranteed to exist for convex or concave functions f.

Assumption 1. *For any n, the cost function $C_n : [0, \infty) \to [0, \infty)$ is convex, continuous, and nondecreasing on $[0, \infty)$, and continuously differentiable on $(0, \infty)$. Furthermore, $C_n(0) = 0$.*

While in many world contexts, cost functions are not convex (because, for example, of fixed costs), the convexity assumption results in a tractable analytical setting that can be used to provide some insights and qualitative results that would be otherwise impossible. This is also presumably the reason why much of the theoretical literature on oligopolistic markets makes this assumption (Novshek, 1985; Kamien et al., 1989; Kamien and Zang, 1991; Varian, 1994; Zhou, 2008; Farahat and Perakis, 2009).

Assumption 2. *The inverse demand function* $p : [0, \infty) \to [0, \infty)$ *is continuous, nonnegative, and nonincreasing, with* $p(0) > 0$*. Its right derivative at* 0 *exists and at every* $q > 0$*, its left and right derivatives also exist.*

Note that some parts of our assumptions are redundant, but are included for easy reference. For example, if $C_n(\cdot)$ is convex and nonnegative, with $C_n(0) = 0$, then it is automatically continuous and nondecreasing.

In a Cournot oligopoly, the maximum possible profit earned by all suppliers is an optimal solution to the following optimization problem,

$$\begin{aligned} &\text{maximize} && p(X) \cdot X - \sum_{n=1}^{N} C_n(x_n) \\ &\text{subject to} && x_n \geq 0, \quad n = 1, \ldots, N, \end{aligned} \tag{1}$$

where $X = \sum_{n=1}^{N} x_n$. We use $\mathbf{x}^P = (x_1^P, \ldots, x_N^P)$ to denote an optimal solution to (1), and let $X^P = \sum_{n=1}^{N} x_n^P$. We will refer to an optimal solution to (1) as a **monopoly output**. For a model with a nonincreasing continuous inverse demand function and continuous convex cost functions, the following assumption guarantees the existence of an optimal solution to (1), because it essentially restricts the optimization to a compact set of vectors $\mathbf{x}$ for which $x_n \leq R$, for all n.

Assumption 3. *There exists some* $R > 0$ *such that* $p(R) \leq \min_n\{C_n'(0)\}$*.*

Under Assumptions 1-3, there must exist an optimal solution to (1). Note however that there may exist multiple optimal solutions to (1), associated with different prices. For example, consider a case where $N = 1$ and the cost function of the single supplier is identically zero. The inverse demand function is

$$p(q) = \begin{cases} -q + 1, & \text{if } 0 \leq q \leq 2/3, \\ \max\{0, -\frac{1}{4}(q - 2/3) + 1/3\}, & \text{if } 2/3 < q. \end{cases}$$

Assumptions 1-3 are satisfied, and it is not hard to see that $x_1 = 1/2$ and $x_1 = 1$ are two monopoly outputs (optimal solutions to the optimization problem (1)), which yield different prices. We define $\mathcal{P}$ as the set of prices resulting from monopoly outputs; that is, a nonnegative real number v belongs to $\mathcal{P}$, if and only if there exists a monopoly output, $\mathbf{x}^P$, with $v = p(X^P)$.

The following assumption guarantees that the objective function in (1) is concave on the interval where it is positive.

Assumption 4. *On the interval where $p(\cdot)$ is positive, the function $p(q)q$ is concave in q.*

Because $p(\cdot)$ is nonincreasing, all concave inverse demand functions satisfy Assumption 4. We observe that many convex inverse demand functions[2] that have been used in oligopoly analysis and marketing research also satisfy Assumption 4. For example, inverse demand functions of the form (Bulow and Pfleiderer, 1983; Corchon, 2008)

$$p(q) = \max\{\alpha - \beta q^{\delta}, 0\}, \qquad \alpha,\ \beta,\ \delta > 0, \tag{2}$$

and the following class of convex inverse demand functions (Bulow and Pfleiderer, 1983; Tyagi, 1999)

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad \alpha,\ \beta > 0, \tag{3}$$

satisfy Assumption 4.

We observe that under Assumptions 1, 2, and 4, the objective function in (1) is concave on the interval where it is positive. Hence, we have the following **necessary and sufficient** conditions for a vector $\mathbf{x}^P$ with $p(X^P) > 0$ to maximize the aggregate profit:

$$\begin{cases} C_n'(x_n^P) \le p\left(X^P\right) + \partial_- p\left(X^P\right) \cdot X^P, & \text{if } x_n > 0, \\ C_n'(x_n^P) \ge p\left(X^P\right) + \partial_+ p\left(X^P\right) \cdot X^P. \end{cases} \tag{4}$$

[2]In general, a Cournot equilibrium need not exist when the inverse demand function is convex. However, it is well known that a Cournot equilibrium will exist if the quantities supplied by different suppliers are strategic substitutes (Bulow et al., 1985; Berry and Pakes, 2003). Existence results for Cournot oligopolies for the case of strategic substitutes can be found in Novshek (1985); Gaudet and Salant (1991). Note however, that the strategic substitutes condition is not necessary for the existence of Cournot equilibria. For example, using the theory of ordinally supermodular games, Amir (1996) shows that the log-concavity of inverse demand functions guarantees the existence of a Cournot equilibrium.

There are similar equilibrium conditions for a strategy profile $\mathbf{x}$. In particular, under Assumptions 1 and 2, if $\mathbf{x}$ is a Cournot equilibrium, then

$$C_n'(x_n) \leq p(X) + x_n \cdot \partial_- p(X), \quad \text{if } x_n > 0, \tag{5}$$
$$C_n'(x_n) \geq p(X) + x_n \cdot \partial_+ p(X), \tag{6}$$

where again $X = \sum_{n=1}^{N} x_n$. Note, however, that in the absence of further assumptions, the payoff of supplier n need not be a concave function of x_n and these conditions are, in general, not sufficient.

We say that a nonnegative vector $\mathbf{x}$ is a **Cournot candidate** if it satisfies the necessary conditions (5)-(6). Note that for a given model, the set of Cournot equilibria is a subset of the set of Cournot candidates. Most of the results obtained in this section apply to all Cournot candidates.

As shown in Friedman (1977), if $p(0) > \min_n\{C_n'(0)\}$, then the aggregate supply at a Cournot equilibrium is positive; see Lemma 1 in Appendix A (Proposition 4 in Tsitsiklis and Xu (2011)) for a slight generalization. If on the other hand $p(0) \leq \min_n\{C_n'(0)\}$, then the model is uninteresting, because no supplier has an incentive to produce and the optimal social welfare is zero. This motivates the assumption that follows.

Assumption 5. *The price at zero supply is larger than the minimum marginal cost of the suppliers, i.e.,*

$$p(0) > \min_n\{C_n'(0)\}.$$

Note that if $N = 1$, a Cournot equilibrium must maximize the aggregate profit. We will therefore study the more interesting case where $N \geq 2$. Given a nonnegative vector $\mathbf{x}$, we define its **profit ratio** $\eta(\mathbf{x})$, by

$$\eta(\mathbf{x}) = \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} C_n(x_n^P)}, \tag{7}$$

where $(x_1^P, \ldots, x_N^P)$ is an optimal solution to the optimization problem (1). Under Assumptions 1-5, the ratio is well defined, because the denominator is positive. According to Lemma 2 in Appendix A, a Cournot candidate yields a nonnegative profit, and therefore its profit ratio is nonnegative. For a Cournot candidate $\mathbf{x}$ with $p(X) = 0$, we must have $\eta(\mathbf{x}) = 0$. In Section 3, we will establish lower bounds on the profit ratio of Cournot candidates that yield positive prices.

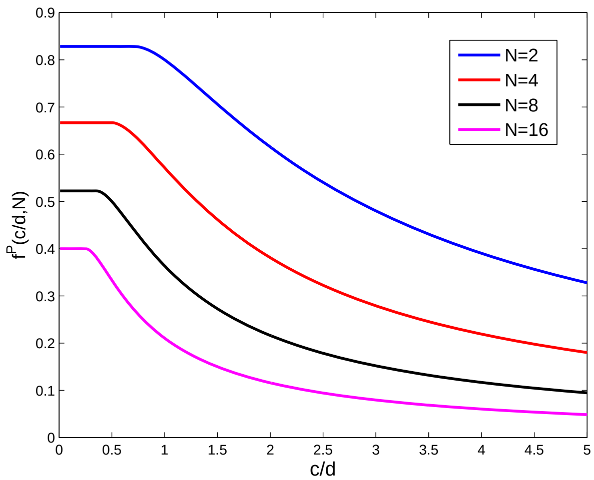

Figure 1: A lower bound on the profit ratio of a Cournot equilibrium as a function of the parameter c/d, for different values of N.

3. Profit ratio lower bounds

We fist establish a lower bound on the profit ratio of a Cournot candidate as a function of the scalar parameter c/d and the number of suppliers (Theorem 1). Through a similar approach, we also provide a lower bound in terms of the scalar parameter c/d, and the maximum of the suppliers' market shares at an equilibrium (Theorem 2).

Theorem 1. *Let* $\mathbf{x}$ *and* $\mathbf{x}^P$ *be a Cournot candidate and a monopoly output, respectively. Suppose that Assumptions 1-5 hold,* $p'(X)$ *exists, and that* $p(X) > 0$*. Let* $c = |(p(X^P) - p(X))/(X^P - X)|$ *and* $d = |p'(X)|$*.*

(a) If $p(X) \in \mathcal{P}$*, then* $\eta(\mathbf{x}) = 1$*;*

(b) If [3] $p(X) \notin \mathcal{P}$*, then* $p'(X) < 0$*. We have* $\eta(\mathbf{x}) \geq f^P(\overline{c}, N)$*, where*

[3]Note that we must have $N \geq 2$; otherwise, a Cournot candidate $\mathbf{x}$ in a model with $N = 1$ satisfies the conditions (5)-(6), which imply the conditions (4). Since $p(X) > 0$, a Cournot candidate for the case $N = 1$ maximizes the aggregate profit and $\mathbf{x} \in \mathcal{P}$.

$\overline{c} = c/d$ *and*

$$f^P(\overline{c}, N) = \begin{cases} \dfrac{N-1+(\sqrt{N}-1)^2}{\sqrt{N}(N-1)}, & \text{if} \quad 0 < \overline{c} \leq \sqrt{1/N}, \\ \dfrac{4\overline{c}^3(N-1)+4\overline{c}(\overline{c}+1)^2}{(\overline{c}^2 N + 2\overline{c}+1)^2}, & \text{if} \quad \overline{c} > \sqrt{1/N}. \end{cases} \tag{8}$$

(c) If $\overline{c} = 1$ (in particular, if $p(\cdot)$ is affine), then $\eta(\mathbf{x}) \geq f^P(1, N) = 4/(N+3)$. Furthermore, the bound is tight, i.e., for any given $N \geq 2$, there exists a model with $\overline{c} = 1$ and a Cournot equilibrium whose profit ratio is $4/(N+3)$.

The theorem is proved in Appendix B. It can be verified that the function $f^P(\overline{c}, N)$ is strictly decreasing in N, as shown in Fig. 1. For any given $\overline{c} > 0$, the lower bound, $f^P(\overline{c}, N)$, decreases to zero as the number of suppliers increases to infinity. Also, for any given N, the profit ratio lower bound is strictly decreasing in $\overline{c}$, over the interval $[\sqrt{1/N}, \infty)$.

Theorem 2. *Let $\mathbf{x}$ and $\mathbf{x}^P$ be a Cournot candidate and a monopoly output, respectively. Suppose that Assumptions 1-5 hold, $p'(X)$ exists, and that $p(X) > 0$. Let $c = |(p(X^P)-p(X))/(X^P-X)|$ and $d = |p'(X)|$. If $p(X) \notin \mathcal{P}$, then $d = |p'(X)| > 0$, and we have:*

(a) $\eta(\mathbf{x}) \geq g^P(\overline{c}, r)$, where $\overline{c} = c/d$, r is the maximum of the suppliers' market shares[4]*, i.e., $r = \max_n\{x_n/X\}$, and*

$$g^P(\overline{c}, r) = \begin{cases} r, & \text{if} \quad 0 < \overline{c} \leq r < 1, \\ \dfrac{4\overline{c}r^2}{(\overline{c}+r)^2}, & \text{if} \quad 0 < r < \overline{c}. \end{cases} \tag{9}$$

(b) If $\overline{c} = 1$ (in particular, if $p(\cdot)$ is affine), then $\eta(\mathbf{x}) \geq g^P(1, N) = 4r^2/(1+r)^2$. Furthermore, the bound is tight. That is, for every $r \in (0, 1)$ and for every $\varepsilon > 0$, there exists a model with $\overline{c} = 1$ and a Cournot equilibrium whose profit ratio is no more than $4r^2/(1+r)^2 + \varepsilon$.

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[4]Lemma 1 shows that $X > 0$, and therefore r is well defined. If $r = 1$, then the Cournot candidate satisfies conditions (4), and therefore maximizes the aggregate profit. Hence, we have $r \in (0, 1)$.

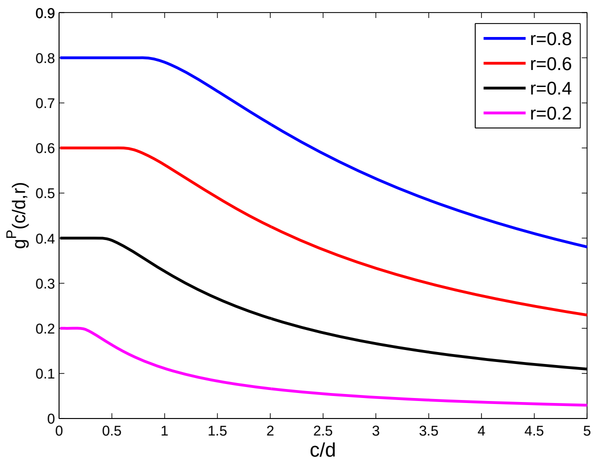

Figure 2: A lower bound on the profit ratio of a Cournot equilibrium as a function of the parameter c/d, for different values of the largest market share r.

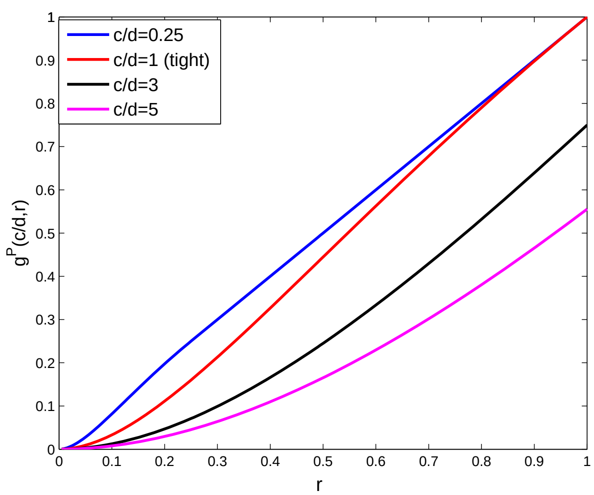

Figure 3: A lower bound on the profit ratio of a Cournot equilibrium as a function of the largest market share r, for different values of the parameter c/d.

The theorem is proved in Appendix C. The derived lower bounds are illustrated in Fig. 2 and Fig. 3. For a given r, the lower bound is nonincreasing in $\overline{c} = c/d$, and for a given $\overline{c}$, the lower bound increases with r. For affine inverse demand functions, we have $\overline{c} = 1$ and the bound is tight (the red curve in Fig. 3).

4. Corollaries and Applications

Given the number of suppliers (or the largest market share at an equilibrium), and the inverse demand function $p(\cdot)$, the lower bounds derived in Theorem 1 (Theorem 2, respectively) require additional knowledge on the aggregate supply at the Cournot equilibrium and on the monopoly output, i.e., X and X^P. For concave inverse demand functions, we first apply Theorems 1 and 2 to establish a lower bound on the profit ratio of a Cournot equilibrium that depends only on the number of suppliers (or on the largest market share at the equilibrium) in Corollary 1. For convex inverse demand functions, in Corollary 2 we establish a profit ratio lower bound that depends only on the inverse demand function, and N (or r). With a small amount information on the supplier cost functions, we further refine the lower bound in Corollary 3. At the end of this section, we apply our results to calculate nontrivial quantitative profit ratio bounds for various inverse demand functions that have been considered in the economics literature.

Corollary 1. *Suppose that Assumptions 1-5 hold, and that $p(\cdot)$ is concave and differentiable on the interval where it is positive. For every Cournot candidate $\mathbf{x}$ with $p(X) > 0$, we have $\eta(\mathbf{x}) \geq \max\{f^P(1, N), g^P(1, r)\}$.*

Proof. If $p(X) \in \mathcal{P}$, then $\eta(\mathbf{x}) = 1$ and the desired result trivially holds. Otherwise, we have that $X^P < X$ and $\overline{c} \leq 1$. The desired result then follows from Theorems 1 and 2, and the fact that both $f^P(\cdot, N)$ and $g^P(\cdot, r)$ are nonincreasing. □

Corollary 2. *Suppose that Assumptions 1-5 hold, and that the inverse demand function $p(\cdot)$ is convex. If $p(Q) = 0$ for some $Q > 0$, and the ratio, $\mu = \partial_+ p(0)/\partial_- p(Q)$, is finite, then for every Cournot candidate $\mathbf{x}$ with $p(X) > 0$, its profit ratio satisfies $\eta(\mathbf{x}) \geq \max\{f^P(\mu, N), g^P(\mu, r)\}$.*

Proof. Since $p(\cdot)$ is nonincreasing and nonnegative, we have $p(q) = 0$ for any $p \geq Q$. Since $p(X) > 0$, we have $X < Q$.

We now argue that $p'(X)$ exists. Lemma 1 shows that $X > 0$. The conditions (5)-(6) applied to some n with $x_n > 0$, imply that

$$p(X) + x_n \cdot \partial_- p(X) \geq p(X) + x_n \cdot \partial_+ p(X).$$

On the other hand, since $p(\cdot)$ is convex, we have $\partial_- p(X) \leq \partial_+ p(X)$. Hence, $\partial_- p(X) = \partial_+ p(X)$, as claimed.

If $p(X) \in \mathcal{P}$, then Proposition 3 implies that $\eta(\mathbf{x}) = 1 \geq f^P(\mu, N)$. Otherwise, since $p(\cdot)$ is convex and $X < Q$, for any aggregate profit maximizing vector, $\mathbf{x}^P$, we have that $\bar{c} \leq \mu$. The desired result follows from Theorems 1 and 2, and the fact that both $f^P(\cdot, N)$ and $g^P(\cdot, r)$ are nonincreasing. $\square$

Corollary 3. *Suppose that Assumptions 1-5 hold, and that $p(\cdot)$ is convex. Let*[5]

$$s = \inf\left\{q \mid p(q) = \min_n C_n'(0)\right\}, \quad t = \inf\left\{q \mid \min_n C_n'(q) \geq p(q) + q\partial_+ p(q)\right\}. \tag{10}$$

If $\partial_- p(s) < 0$, then the profit ratio of a Cournot candidate $\mathbf{x}$ with $p(X) > 0$ is at least

$$\max\{f^P\left(\partial_+ p(t)/\partial_- p(s), N\right), \ g^P\left(\partial_+ p(t)/\partial_- p(s), r\right)\}.$$

The proof of Corollary 3 is given in Appendix D. If there exists a "best" supplier n such that $C_n'(x) \leq C_m'(x)$, for any other supplier m and any $x > 0$, then the parameters s and t depend only on $p(\cdot)$ and $C_n'(\cdot)$. In the following three examples, we apply Corollary 3 to three forms of convex inverse demand functions that appear in the economics literature.

Example 1. Suppose that Assumptions 1, 3, and 5 hold. Among the $N \geq 2$ suppliers, there is a best supplier that has a linear cost function with a slope $\chi \geq 0$. Consider an inverse demand function of the form in (3):

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad \alpha, \ \beta > 0.$$

Note that Corollary 2 does not apply, because the left derivative of $p(\cdot)$ at 0

[5]Under Assumption 3, the existence of the real numbers defined in (10) is guaranteed, and $t \leq s$.

is infinite[6]. Since

$$\frac{d^2(qp(q))}{dq^2} = 2p'(q) + qp''(q) = \frac{-2\beta}{q} + \frac{q\beta}{q^2} < 0, \qquad \forall q \in (0, \exp(\alpha/\beta)),$$

we know that Assumption 4 holds. Through a simple calculation we have

$$s = \exp\left(\frac{\alpha - \chi}{\beta}\right), \qquad t = \exp\left((\alpha - \beta - \chi)/\beta\right).$$

We also have

$$\frac{p'(t)}{p'(s)} = \frac{\exp\left((\alpha - \chi)/\beta\right)}{\exp\left((\alpha - \beta - \chi)/\beta\right)} = \exp(1),$$

and Corollary 3 implies that for every Cournot equilibrium $\mathbf{x}$ with $p(X) > 0$,

$$\eta(\mathbf{x}) \geq \max\{f^P(\exp(1), N),\ g^P(\exp(1), r)\}. \tag{11}$$

Now we argue that the lower bound (11) holds even without the assumption that there is a best supplier associated with a linear cost function. From Proposition 1, the profit ratio of any Cournot equilibrium $\mathbf{x}$ will not increase if the cost function of each supplier n is replaced by

$$\overline{C}_n(x) = C'_n(x_n)x, \qquad \forall x \geq 0.$$

Let $c = \min_n\{C'_n(x_n)\}$. Since the profit ratio lower bound in (11) holds for the modified model with linear cost functions, it applies whenever the inverse demand function is of the form (3). □

Example 2. Suppose that Assumption 1, 3, and 5 hold. There are $N \geq 2$ suppliers. There exists a best supplier, the cost function of which is linear with a slope $\chi \geq 0$. Consider inverse demand functions of the form in Eq. (2):

$$p(q) = \max\{\alpha - \beta q^{\delta}, 0\}, \qquad \alpha,\ \beta,\ \delta > 0.$$

[6] In fact, $p(0)$ is undefined. This turns out to not be an issue: for a small enough $\epsilon > 0$, at a monopoly output and at a Cournot equilibrium, we can guarantee that no supplier chooses a quantity below ϵ. For this reason, the details of the inverse demand function in the vicinity of zero are immaterial as far as the chosen quantities or the resulting aggregate profit are concerned. A similar argument also applies to Example 3.

It is not hard to see that Assumption 2 holds. Assumption 5 implies that $\alpha > \chi$. Since

$$\frac{d^2(qp(q))}{dq^2} \le p'(q) + qp''(q) = -\beta\delta q^{\delta-1} - \beta\delta(\delta-1)q^{\delta-1} = -\beta\delta^2 q^{\delta-1} \le 0,$$

we know that Assumption 4 holds. Through a simple calculation we have

$$s = \left(\frac{\alpha-\chi}{\beta}\right)^{1/\delta}, \quad t = \left(\frac{\alpha-\chi}{\beta(\delta+1)}\right)^{1/\delta}.$$

We also have

$$\frac{p'(t)}{p'(s)} = \frac{-\beta\delta t^{\delta-1}}{-\beta\delta s^{\delta-1}} = (\delta+1)^{\frac{1-\delta}{\delta}}.$$

From Corollary 3 we know that for every Cournot equilibrium $\mathbf{x}$ with $p(X) > 0$,

$$\eta(\mathbf{x}) \ge \max\left\{ f^P((\delta+1)^{\frac{1-\delta}{\delta}}, N),\ g^P((\delta+1)^{\frac{1-\delta}{\delta}}, r)\right\}.$$

By the same argument in Example 1, we know that the derived profit ratio lower bound holds for general cost functions, as long as the inverse demand function is of the form in Eq. (2). □

Example 3. Suppose that Assumptions 1, 3, and 5 hold. Among the $N \ge 2$ suppliers, there is a best supplier that has a linear cost function with a slope $\chi \ge 0$. Consider constant elasticity inverse demand functions, of the form (cf. Eq. (4) in Bulow and Pfleiderer (1983))

$$p(q) = \alpha q^{-\beta}, \qquad 0 \le \alpha, \qquad 0 \le \beta < 1. \tag{12}$$

Assumption 5 implies that $\alpha > \chi$. Since

$$\frac{d^2(qp(q))}{dq^2} = 2p'(q) + qp''(q) = -\alpha\beta(1-\beta)q^{-\beta-1} \le 0,$$

we know that Assumption 4 holds. Through a simple calculation we have,

$$s = \left(\frac{\chi}{\alpha}\right)^{-1/\beta}, \qquad t = \left(\frac{\chi}{\alpha(1-\beta)}\right)^{-1/\beta}.$$

We have

$$\frac{p'(t)}{p'(s)} = \frac{-\alpha\beta t^{-\beta-1}}{-\alpha\beta s^{-\beta-1}} = (1-\beta)^{\frac{-\beta-1}{\beta}}.$$

From Corollary 3 we conclude that a Cournot equilibrium $\mathbf{x}$ with $p(X) > 0$ must satisfy

$$\eta(\mathbf{x}) \geq \max \left\{ f^P((1-\beta)^{\frac{-\beta-1}{\beta}}, N),\ g^P((1-\beta)^{\frac{-\beta-1}{\beta}}, r) \right\}.$$

Following the argument in the end of Example 1, we conclude that the lower bound on the profit ratio holds for general cost functions, as long as the inverse demand function is of the form in Eq. (12). $\square$

5. Conclusion

For Cournot oligopoly models with concave revenue functions, results such as those provided in Theorem 2 (or Theorem 1) show that the profit ratio at a Cournot equilibrium can be lower bounded by a function of the largest market share at the equilibrium (or the number of suppliers), and a scalar parameter that captures quantitative properties of the inverse demand function. Our results allow us to lower bound the profit ratio of Cournot equilibria for a large class of Cournot oligopoly models that have been considered in the economics literature, without having to restrict to the special case of affine demand functions, and without having to calculate the equilibrium and the profit-maximizing output.

Our results suggest that the degree of nonlinearity of the inverse demand function and the largest market share at an equilibrium (or the number of suppliers) may have a significant impact on the profit loss at an equilibrium. As the number of suppliers increases to infinity (or the largest market share shrinks to zero), the profit ratio lower bound converges to zero and arbitrarily high profit loss is possible. For an oligopolistic market with a small number of suppliers, N (or with a large supplier that holds a considerable fraction of market share, r), the established profit ratio lower bounds depend on the scalar parameter derived from the inverse demand function. For the important special class of Cournot models with affine inverse demand functions, our profit ratio lower bounds, $4/(N+3)$ and $4r^2/(1+r)^2$, are tight.

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Appendix A. Preliminaries on Cournot Equilibria

Lemma 1. *Suppose that Assumptions 1, 2, and 5 hold. If* $\mathbf{x}$ *is a Cournot candidate, then* $X > 0$.

Proof. Suppose that $p(0) > \min_n\{C_n'(0)\}$. Then, the vector $\mathbf{x} = (0, \ldots, 0)$ violates condition (6), and cannot be a Cournot candidate. $\square$

Lemma 2. *Suppose that Assumptions 1-5 hold. Then, the aggregate profit achieved at a Cournot candidate is nonnegative, and the optimal objective value of the optimization problem (1) is positive.*

Proof. Let $\mathbf{x}$ be a Cournot candidate. Lemma 1 shows that $X > 0$. For every supplier n such that $x_n > 0$, according to the necessary condition (5), we have $C_n'(x_n) \leq p(X)$. Hence,

$$\sum_{n=1}^{N} C_n'(x_n)x_n \leq p(X) \cdot \sum_{n=1}^{N} x_n.$$

Since each $C_n(\cdot)$ is convex and nondecreasing, we have

$$\sum_{n=1}^{N} C_n(x_n) \leq \sum_{n=1}^{N} C_n'(x_n)x_n \leq p(X) \cdot X. \tag{A.1}$$

Hence, the aggregate profit achieved at the Cournot candidate, $Xp(X) - \sum_{n=1}^{N} C_n(x_n)$, is nonnegative. Let

$$k \in \arg\min_n \{C_n'(0)\}.$$

Due to Assumption 5 and the continuity of the inverse demand and cost functions, there exists some $\varepsilon > 0$ such that

$$\varepsilon p(\varepsilon) - C_k(\varepsilon) > 0,$$

which implies that the optimal objective value in the optimization problem (1) is positive. $\square$

In the following proposition, we show that in order to study the worst-case profit ratio of Cournot equilibria, it suffices to consider linear cost functions. This is a counterpart of (but different from) Proposition 6 in Tsitsiklis and Xu (2011).

Proposition 1. *Suppose that Assumptions 1-5 hold. Let* $\mathbf{x}$ *be a Cournot candidate that is not an optimal solution to (1), and let* $\alpha_n = C'_n(x_n)$*. Consider a modified model in which we replace the cost function of each supplier* n *by a new function* $\overline{C}_n(\cdot)$*, defined by*

$$\overline{C}_n(x) = \alpha_n x, \qquad \forall x \geq 0.$$

Then, for the modified model, Assumptions 1-5 still hold, the vector $\mathbf{x}$ *is a Cournot candidate and its profit ratio, denoted by* $\overline{\eta}(\mathbf{x})$*, satisfies* $0 \leq \overline{\eta}(\mathbf{x}) \leq \eta(\mathbf{x})$*.*

Proof. We first observe that the vector $\mathbf{x}$ satisfies the necessary conditions (5)-(6) for the modified model. Hence, the vector $\mathbf{x}$ is a Cournot candidate in the modified model. It is not hard to see that Assumptions 1, 2 and 4 hold in the modified model. Finally, since $\alpha_n \geq C'_n(0)$ for every n, Assumption 3 holds in the modified model.

We now show that Assumption 5 holds in the modified model, i.e., that $p(0) > \min_n\{\alpha_n\}$. Since the vector $\mathbf{x}$ is a Cournot candidate in the original model, Lemma 1 implies that $X > 0$. Consider a supplier n such that $x_n > 0$. From the necessary conditions (5), we have that $\alpha_n \leq p(X)$. If $\alpha_n = p(X) = 0$, then $p(0) > 0 = \min_n\{\alpha_n\}$. If $\alpha_n = p(X) > 0$, then $\partial_- p(X) = 0$. Since $\partial_+ p(X) \leq 0$ and $x_n \leq X$, the necessary conditions (5)-(6) imply the conditions in (4), which are sufficient for $\mathbf{x}$ to maximize the aggregate profit. Since $\mathbf{x}$ is not an optimal solution to the profit maximization problem (1), it follows that $\alpha_n < p(X)$, which implies that Assumption 5 holds in the modified model.

Let $\mathbf{x}^P$ be an optimal solution to (1) in the original model. Lemma 2 implies that $p(X^P) > 0$. Since $\mathbf{x}^P$ satisfies the conditions in (4) for the modified model, it remains a monopoly output in the modified model. In the modified model, since Assumptions 1-5 hold, the profit ratio of the vector $\mathbf{x}$ is well defined and given by

$$\overline{\eta}(\mathbf{x}) = \frac{Xp(X) - \sum_{n=1}^{N} \alpha_n x_n}{X^P p(X^P) - \sum_{n=1}^{N} \alpha_n x_n^P}. \tag{A.2}$$

Note that the denominator on the right-hand side of (A.2) is the maximum aggregate profit in the modified model, and the numerator is the aggregate profit achieved at the Cournot candidate $\mathbf{x}$ in the modified model. Lemma 2 shows that the denominator is positive, while the numerator is nonnegative.

Since $C_n(\cdot)$ is convex, we have

$$C_n(x_n^P) \geq C_n(x_n) + \alpha_n(x_n^P - x_n), \qquad n = 1, \ldots, N.$$

Adding a nonnegative quantity to the denominator cannot increase the ratio and, therefore,

$$\eta(\mathbf{x}) = \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} C_n(x_n^P)} \geq \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} \left(\alpha_n(x_n^P - x_n) + C_n(x_n)\right)}. \tag{A.3}$$

Note that the right-hand side of (A.3) is nonnegative because $\mathbf{x}$ is a Cournot candidate in the original model (cf. Lemma 2). Since $C_n(\cdot)$ is convex and nondecreasing, with $C_n(0) = 0$, we have

$$\sum_{n=1}^{N} C_n(x_n) - \sum_{n=1}^{N} \alpha_n x_n \leq 0. \tag{A.4}$$

Since the right-hand side of (A.3) is in the interval $[0, 1]$, adding a nonpositive quantity to both the numerator and the denominator cannot increase the ratio. Therefore, using (A.4) in the first inequality below we have

$$\begin{aligned}
&\eta(\mathbf{x}) \\
\geq &\frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} \left(\alpha_n(x_n^P - x_n) + C_n(x_n)\right)} \\
\geq &\frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n) + \left(\sum_{n=1}^{N} C_n(x_n) - \sum_{n=1}^{N} \alpha_n x_n\right)}{X^P p(X^P) - \sum_{n=1}^{N} \left(\alpha_n(x_n^P - x_n) + C_n(x_n)\right) + \left(\sum_{n=1}^{N} C_n(x_n) - \sum_{n=1}^{N} \alpha_n x_n\right)} \\
= &\frac{Xp(X) - \sum_{n=1}^{N} \alpha_n x_n}{X^P p(X^P) - \sum_{n=1}^{N} \alpha_n x_n^P} \\
= &\overline{\eta}(\mathbf{x}).
\end{aligned}$$

The desired result follows. □

If $\mathbf{x}$ is a Cournot equilibrium, then it satisfies Eqs. (5)-(6), and therefore is a Cournot candidate. Hence, Proposition 1 applies to all Cournot equilibria that do not maximize the aggregate profit. We note that if a Cournot equilibrium $\mathbf{x}$ maximizes the aggregate profit for the original model, then the maximum aggregate profit in the modified model could be zero, in which case $\eta(\mathbf{x}) = 1$, but $\overline{\eta}(\mathbf{x})$ is undefined; see the example that follows.

Example 4. Consider a model involving a single supplier ($N = 1$). The cost function of supplier 1 is $C_1(x) = x^2$. The inverse demand function is constant, with $p(q) = 1$ for any $q \geq 0$. It is not hard to see that the vector $x_1 = 1/2$ is a Cournot equilibrium, which also maximizes the aggregate profit. In the modified model, we have $\overline{C}_1(x) = x$. The aggregate profit achieved in the modified model is always zero, regardless of the action taken by the supplier. $\square$

Proposition 1 shows that a Cournot candidate in the original model remains a Cournot candidate in the modified model. Hence, to lower bound the profit ratio of a Cournot equilibrium in the original model, it suffices to lower bound the profit ratio of a worst Cournot candidate for a modified model. Accordingly, and for the purpose of deriving lower bounds, we can (and will) restrict to the case of linear cost functions, and study the worst case profit ratio over all Cournot candidates.

In the following two propositions, we show that at a Cournot candidate there are two possibilities: either $p(X^P) > p(X)$ and $X^P < X$, or $p(X) \in \mathcal{P}$ (Proposition 2); in the latter case, under the additional assumption that $p'(X)$ exists, a Cournot candidate maximizes the aggregate profit (Proposition 3). We then show in Proposition 4 that to lower bound the worst case profit ratio, it suffices to restrict to a special class of piecewise linear inverse demand functions.

Proposition 2. *Suppose that Assumptions 1-5 hold. Let* $\mathbf{x}$ *a Cournot candidate. If* $p(X) \notin \mathcal{P}$, *then for any optimal solution* $\mathbf{x}^P$ *to (1), we have* $X > X^P$.

Proof. Suppose that $p(X) \in \mathcal{P}$. Lemma 2 implies that $p(X^P) > 0$ for every monopoly output $\mathbf{x}^P$. If $p(X) = 0$, then we know that $X > X^P$, because $p(\cdot)$ is nonincreasing.

Now consider the case where $p(X) > 0$. Lemma 1 shows that $X > 0$. If there is only one supplier that provides a positive quantity at the Cournot candidate, then the necessary conditions (5)-(6) imply the conditions in (4), and we conclude that the Cournot candidate maximizes the aggregate profit, i.e., $p(X) \in \mathcal{P}$, a contradiction. Hence, there are at least two suppliers who produce positive quantities at the Cournot candidate. We therefore have that $X > x_n$, for any $n = 1, \ldots, N$.

Suppose that there exists an optimal solution to (1), $\mathbf{x}^P$, such that $0 < X \leq X^P$. Since $p(X) \neq p(X^P)$ and $p(\cdot)$ is nonincreasing, we have $X^P > X$

and $p(X^P) < p(X)$. For every supplier n for which $x_n^P > 0$, we have

$$C_n'(x_n^P) \leq \partial_- p(X^P)X^P + p(X^P) \leq \partial_+ p(X)X + p(X) \leq \partial_+ p(X)x_n + p(X), \tag{A.5}$$

where the first inequality follows from (4), the second inequality follows from the fact $X < X^P$ and Assumption 4, and the last inequality holds because $\partial_+ p(X) \leq 0$ and $x_n < X$. We now argue that equality cannot hold throughout (A.5). If $\partial_+ p(X) < 0$, then the last inequality is strict; if $\partial_+ p(X) = 0$, the second inequality is strict because $p(X^P) < p(X)$ and $\partial_- p(X) \leq 0$. Using also the necessary condition (6), we have

$$C_n'(x_n^P) < \partial_+ p(X)x_n + p(X) \leq C_n'(x_n).$$

Due to the convexity of the cost functions, it follows that $x_n > x_n^P$ for every n such that $x_n^P > 0$, which contradicts our hypothesis that $X \leq X^P$. □

If the inverse demand function does not satisfy Assumption 4, it is possible that the aggregate supply at a Cournot candidate is less than that at a monopoly output, as shown in the following example.

Example 5. Consider a model involving only one supplier ($N = 1$). The cost function of the supplier is linear with a slope of 2. The inverse demand function is given by

$$p(q) = \begin{cases} -q + 4, & \text{if} \quad 0 \leq q \leq 4/3, \\ \max\{0, -\frac{1}{5}(q - 4/3) + 8/3\}, & \text{if} \quad 4/3 < q, \end{cases}$$

which satisfies Assumption 2. It can be verified that the supplier can maximize its profit at $x_1 = 7/3$. It is also easy to check that $x_1 = 1$ is a Cournot candidate. We have $X = 1 < 7/3 = X^P$. □

Proposition 3. *Suppose that Assumptions 1-5 hold. Let $\mathbf{x}$ be a Cournot candidate. If $p(X) \in \mathcal{P}$ and $p'(X)$ exists, then $\eta(\mathbf{x}) = 1$.*

Proof. From Lemma 2 we have that $p(X^P) > 0$ for every monopoly output $\mathbf{x}^P$. Since $p(X) \in \mathcal{P}$, it follows that $p(X) > 0$. Lemma 1 implies that $X > 0$. If there is only one supplier n who provides a positive quantity of good at the Cournot candidate, then the necessary conditions (5)-(6) imply the conditions in (4), and we conclude that $\eta(\mathbf{x}) = 1$.

Now consider the case where $\mathbf{x}$ has at least two positive components. Then, $X > x_n$, for any $n = 1, \ldots, N$. Since $p(X) \in \mathcal{P}$, there exists an optimal solution to (1), $\mathbf{x}^P$, such that $p(X) = p(X^P)$. We now prove that $p'(X) = 0$. Suppose not, in which case we have $p'(X) < 0$. Since $p(X) = p(X^P)$, we have that $X = X^P$. For every n such that $x_n^P > 0$, from the convexity of $C_n(\cdot)$ and the conditions in (4), we have

$$C_n'(0) \leq C_n'(x_n^P) = p'\big(X^P\big) \cdot X^P + p(X^P).$$

Since $X = X^P$, $p'(X) < 0$, and $x_n < X$, we have

$$C_n'(0) \leq p'\big(X^P\big) \cdot X^P + p(X^P) = p'(X) \cdot X + p(X) < p'(X) \cdot x_n + p(X),$$

which implies that $x_n > 0$, from the necessary condition (6). Hence, we have

$$C_n'(x_n^P) = p'\big(X^P\big) \cdot X^P + p(X^P) < p'(X) \cdot x_n + p(X) = C_n'(x_n),$$

which implies that $x_n > x_n^P$, from the convexity of $C_n(\cdot)$. Since $x_n > x_n^P$ for every n such that $x_n^P > 0$, we conclude that $X > X^P$, which contradicts our earlier conclusion that $X = X^P$. The contradiction shows that $p'(X) = 0$.

Since $p'(X) = 0$ and the Cournot candidate $\mathbf{x}$ satisfies the necessary conditions (5)-(6), it also satisfies the conditions in (4). Furthermore, since $p(X) > 0$, the conditions in (4) are sufficient for the Cournot candidate $\mathbf{x}$ to maximize the aggregate profit, i.e., $\eta(\mathbf{x}) = 1$. □

It can be shown that if the inverse demand function is convex, then $p'(X)$ exists for any Cournot candidate $\mathbf{x}$ (cf. Proposition 3 in Tsitsiklis and Xu (2011)). On the other hand, for a model satisfying Assumptions 1-5, if the inverse demand function is not differentiable at X, then a Cournot equilibrium $\mathbf{x}$ may yield arbitrarily large profit loss, even if $X = X^P$, and N is held fixed.

Example 6. Consider a model involving two suppliers ($N = 2$), with $C_1(x) = 0$ and $C_2(x) = x$. The inverse demand function is concave on the interval where it is positive, of the form

$$p(q) = \begin{cases} 1, & \text{if } \ 0 \leq q \leq 1, \\ \max\{0, -M(q-1)+1\}, & \text{if } \ 1 < q, \end{cases}$$

where $M > 2$. At the vector $(1, 0)$, the maximum aggregate profit, 1, is achieved. The aggregate profit realized at the Cournot equilibrium $\mathbf{x} = (1/M, 1 - 1/M)$ is $1/M$. Note that $X = X^P = 1$. However, the profit ratio of $\mathbf{x}$ can be made arbitrarily small, as M grows large. □

Based on the preceding propositions, we are now ready to prove the following proposition, which will serve as a basis for the main theorems to be given in the next section.

Proposition 4. *Let* $\mathbf{x}$ *and* $\mathbf{x}^P$ *be a Cournot candidate and a monopoly output, respectively. Suppose that Assumptions 1-5 hold,* $p'(X)$ *exists, and that* $p(X) \notin \mathcal{P}$. *Let* $c = |(p(X^P) - p(X))/(X^P - X)|$, $d = |p'(X)|$. *Now consider a modified model in which the inverse demand function is replaced by a piecewise linear function*[7] $p^0(\cdot)$,

$$p^0(q) = \begin{cases} -c(q - X) + p(X), & 0 \leq q \leq X, \\ \max\{0, -d(q - X) + p(X)\}, & X < q. \end{cases} \tag{A.6}$$

Let $\eta^0(\mathbf{x})$ *be the profit ratio of the vector* $\mathbf{x}$ *in the modified model. We have*

$$\eta^0(\mathbf{x}) \leq \eta(\mathbf{x}).$$

Proof. Since $p^0(X) = p(X)$, the aggregate profit earned at $\mathbf{x}$ is

$$Xp(X) - \sum\nolimits_{n=1}^{N} C_n(x_n),$$

in both the original and the modified model. Hence, we have

$$\eta^0(\mathbf{x}) \leq \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p^0(X^P) - \sum_{n=1}^{N} C_n(x_n^P)} = \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} C_n(x_n^P)} = \eta(\mathbf{x}),$$

where the inequality holds because the maximum total profit in the modified model is at least $X^P p^0(X^P) - \sum_{n=1}^{N} C_n(x_n^P)$, and the next equality holds because $p^0(X^P) = p(X^P)$. □

Proposition 4 shows that a lower bound on the profit ratio of a Cournot equilibrium can be established by calculating its profit ratio in a modified model with a piecewise linear inverse demand function. This result enables us to derive our main results, given in the next section.

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[7] According to Proposition 2, we have $X^P < X$. The first segment of the piecewise linear function $p^0(\cdot)$ agrees with the inverse demand function $p(\cdot)$ at the two points: $(X^P, p(X^P))$ and $(X, p(X))$; the second segment is tangent to the inverse demand curve $p(\cdot)$ at the point $(X, p(X))$.

Appendix B. Proof of Theorem 1

According to Proposition 3, if $p(X) \in \mathcal{P}$, then the Cournot candidate's profit ratio must equal one.

To prove part (b), we assume that $p(X) \notin \mathcal{P}$. If $p'(X) = 0$, the necessary conditions (5)-(6) imply the conditions in (4). Since $p(X) > 0$, the conditions in (4) are sufficient for $\mathbf{x}$ to maximize the aggregate profit. But since $p(X) \notin \mathcal{P}$, this cannot be the case and we must have $p'(X) < 0$ and $d > 0$.

We have shown in Proposition 1 that the vector $\mathbf{x}$ remains a Cournot candidate in the modified model with linear cost functions, and Assumptions 1-5 still hold. Further, to lower bound the worst case profit ratio for Cournot candidates, we only need to derive a lower bound for the profit ratio of Cournot candidates for the case of linear cost functions. We therefore assume that $C_n(x_n) = \alpha_n x_n$ for each n. Without loss of generality, we further assume that $\alpha_1 = \min_n\{\alpha_n\}$.

Since $p'(X)$ exists, we have the following necessary (and, by definition, sufficient) conditions for a nonzero vector $\mathbf{x}$ to be a Cournot candidate:

$$\begin{cases} C_n'(x_n) = p\,(X) + x_n p'(X), & \text{if } x_n > 0, \\ C_n'(0) \geq p\,(X) + x_n p'(X), & \text{if } x_n = 0. \end{cases} \tag{B.1}$$

Since $p(X) \neq p(X^P)$, we have that $c > 0$. For conciseness, we let $y = p(X)$ throughout the proof. We will prove the theorem by considering separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$. According to Proposition 4, the profit ratio of the Cournot candidate $\mathbf{x}$ is lower bounded by the profit ratio $\eta^0(\mathbf{x})$ for the case of a piecewise linear function of the form in (A.6).

The case $\alpha_1 = 0$

Let $\mathbf{x}$ be a Cournot candidate in the original model, with linear cost functions and the inverse demand function $p(\cdot)$. Suppose first that $x_1 = 0$. The second inequality in (B.1), with $n = 1$, and $C_n'(0) = \alpha_1 = 0$, implies that $p(X) = 0$. Since $p(X) > 0$, we must have $x_1 > 0$. The first equality in (B.1) yields $y > 0$ and $x_1 = y/d$. We have

$$0 \leq \sum_{n=2}^{N} x_n = X - \frac{y}{d}, \tag{B.2}$$

from which we conclude that $X \geq y/d$.

From Proposition 4, the profit ratio of the vector $\mathbf{x}$ in the modified model, $\eta^0(\mathbf{x})$, cannot be more than its profit ratio in the original model, $\eta(\mathbf{x})$. Hence, to prove part (b), it suffices to show that $\eta^0(\mathbf{x}) \geq f^P(\bar{c}, N)$. For the modified model, the maximum aggregate profit is the optimal value of the following optimization problem,

$$\begin{aligned} &\text{maximize} \;\; qp^0(q) \\ &\text{subject to} \;\; q \geq 0. \end{aligned}$$

Since $dX \geq y$, the derivative of the aggregate profit is nonpositive at $q = X$, and so the aggregate profit is nonincreasing with q on the interval $[X, \infty)$. Hence, in the modified model, the aggregate profit is maximized in the interval $[0, X]$. Through a simple calculation we have:

(i) If $cX \geq y$, then the maximum aggregate profit is $(cX+y)^2/(4c)$, achieved at $q = (cX + y)/(2c)$.

(ii) If $cX \leq y$, then the maximum aggregate profit is Xy, achieved at $q = X$.

Note that for $n = 1$ we have $\alpha_n x_n = 0$. For $n \geq 2$, whenever $x_n > 0$, from the first equality in (B.1) we have $\alpha_n = y - x_n d$ and $\alpha_n x_n = (y - x_n d)x_n$. Since $\alpha_n \geq 0$, we have $y \geq x_n d$, for $n = 2, \ldots, N$. Therefore,

$$(N-1)y \geq d \sum_{n=2}^{N} x_n = dX - y,$$

i.e.,

$$Ny \geq dX. \tag{B.3}$$

In the modified model, the aggregate profit achieved at $\mathbf{x}$ is

$$\begin{aligned} Xp(X) - \sum_{n=1}^{N} \alpha_n x_n &= Xy - \sum_{n=2}^{N} (y - x_n d)x_n \\ &\geq Xy - y\sum_{n=2}^{N} x_n + \frac{(Xd-y)^2}{(N-1)d} \\ &= Xy - y(X - y/d) + \frac{(Xd-y)^2}{(N-1)d} \\ &= \frac{y^2}{d} + \frac{(Xd-y)^2}{(N-1)d}, \end{aligned} \tag{B.4}$$

where the inequality is true because $\sum_{n=2}^{N} x_n^2$ is minimized when $x_2 = x_3 = \cdots = x_N$, subject to the constraint in (B.2). For the case $cX \geq y$, since the maximum aggregate profit is $(cX+y)^2/4c$, we have

$$\eta^0(\mathbf{x}) \geq \frac{y^2/d + (Xd-y)^2/((N-1)d)}{(cX+y)^2/4c}. \tag{B.5}$$

Note that c, d, and y are positive. Substituting $\overline{y} = cX/y$ and $\overline{c} = c/d$ to (B.5), we have

$$\eta^0(\mathbf{x}) \geq \frac{4\overline{c}^2 + 4(\overline{y}-\overline{c})^2/(N-1)}{\overline{c}(\overline{y}+1)^2}, \quad 1 \leq \overline{y} \leq N\overline{c}, \quad 0 < \overline{c} \leq \overline{y}, \tag{B.6}$$

where the constraints $\overline{y} \geq \overline{c}$ and $\overline{y} \leq N\overline{c}$ follow from (B.2) and (B.3), respectively. For any given $\overline{c} \geq 1/N$, through a simple calculation we obtain that the right-hand side of the first inequality in (B.6) is minimized at

$$\overline{y} = \max\left\{\frac{\overline{c}^2 N + \overline{c}}{\overline{c}+1}, 1\right\}.$$

We conclude that if $cX \geq y$, then

$$\eta^0(\mathbf{x}) \geq \begin{cases} \dfrac{\overline{c}^2 + (1-\overline{c})^2/(N-1)}{\overline{c}}, & \text{if} \quad 1/N \leq \overline{c} \leq \sqrt{1/N}, \\ \dfrac{4\overline{c}^3(N-1) + 4\overline{c}(\overline{c}+1)^2}{(\overline{c}^2 N + 2\overline{c}+1)^2}, & \text{if} \quad \overline{c} > \sqrt{1/N}. \end{cases} \tag{B.7}$$

Note that the case $\overline{c} < 1/N$ cannot happen, by (B.6).

For the case $cX \leq y$, the maximum aggregate profit is Xy. From (B.4), we have

$$\eta^0(\mathbf{x}) \geq \frac{\dfrac{y^2}{d} + \dfrac{(Xd-y)^2}{(N-1)d}}{Xy} = \frac{\overline{c}^2(N-1) + (\overline{y}-\overline{c})^2}{\overline{c}\cdot\overline{y}(N-1)}, \quad 0 < \overline{c} \leq \overline{y} \leq 1, \quad \overline{y} \leq N\overline{c}, \tag{B.8}$$

where the constraints $\overline{y} \geq \overline{c}$ and $\overline{y} \leq N\overline{c}$ follow from (B.2) and (B.3), respectively. Through a simple calculation, for any given $\overline{c} \in (0, 1]$, we find that the right-hand side of (B.8) is minimized at

$$\overline{y} = \min\{\overline{c}\sqrt{N}, 1\}.$$

We conclude that if $cX \le y$, then

$$\eta^0(\mathbf{x}) \ge \begin{cases} \dfrac{N-1+(\sqrt{N}-1)^2}{\sqrt{N}(N-1)}, & \text{if} \quad 0 < \overline{c} \le \sqrt{1/N}, \\ \dfrac{\overline{c}^2+(1-\overline{c})^2/(N-1)}{\overline{c}}, & \text{if} \quad \sqrt{1/N} \le \overline{c} \le 1. \end{cases} \tag{B.9}$$

Through another simple calculation, we conclude that

$$\frac{\overline{c}^2+(1-\overline{c})^2/(N-1)}{\overline{c}} \ge \frac{(N-1)+(\sqrt{N}-1)^2}{\sqrt{N}(N-1)}, \qquad \text{if} \;\; 0 < \overline{c} \le \sqrt{1/N}, \tag{B.10}$$

because the left-hand side is nonincreasing in $\overline{c}$, and the left-hand side equals the right-hand side if $\overline{c} = \sqrt{1/N}$. Similarly,

$$\frac{\overline{c}^2+(1-\overline{c})^2/(N-1)}{\overline{c}} \ge \frac{4\overline{c}^3(N-1)+4\overline{c}(\overline{c}+1)^2}{(\overline{c}^2N+2\overline{c}+1)^2}, \qquad \text{if} \;\; \overline{c} \ge \sqrt{1/N}, \tag{B.11}$$

because the left-hand side equals the right-hand side at $\overline{c} = \sqrt{1/N}$, and the derivative of the left-hand side, with respect to $\overline{c}$, is more than that of the right-hand side, for every $\overline{c} \ge \sqrt{1/N}$. Combining the results in (B.7) and (B.9)-(B.11), we have

$$\eta^0(\mathbf{x}) \ge \begin{cases} \dfrac{N-1+(\sqrt{N}-1)^2}{\sqrt{N}(N-1)}, & \text{if} \quad 0 < \overline{c} \le \sqrt{1/N}, \\ \dfrac{4\overline{c}^3(N-1)+4\overline{c}(\overline{c}+1)^2}{(\overline{c}^2N+2\overline{c}+1)^2}, & \text{if} \quad \overline{c} > \sqrt{1/N}. \end{cases} \tag{B.12}$$

Tightness

Given an integer $N \ge 2$, consider a model with an affine inverse demand function $p^0(\cdot)$ of the form (A.6), with $c/d = 1$ and $cX \ge y > 0$. Let the cost of supplier 1 be identically zero and let

$$C_n(x) = \left(y - \frac{d}{N-1}\left(X - \frac{y}{d}\right)\right)x, \qquad n = 2, \ldots, N. \tag{B.13}$$

It is not hard to see that the vector with components

$$x_1 = \frac{y}{d}, \qquad x_n = \frac{1}{N-1}\Big(X - \frac{y}{d}\Big), \qquad n = 2, \ldots, N, \tag{B.14}$$

satisfies the conditions (5)-(6). It can be verified that $\mathbf{x}$ is a Cournot equilibrium.

For the case where $cX \geq y$ and $\min_n\{C_n'(\cdot)\} = 0$, we have shown that the maximum total profit is $(cX+y)^2/4c$, and the aggregate profit achieved at $\mathbf{x}$ is given by the right-hand side of (B.4). When $\overline{y} = cX/y = (N+1)/2$, the profit ratio of the Cournot equilibrium $\mathbf{x}$ is given by

$$\eta(\mathbf{x}) = \frac{y^2/d + (Xd-y)^2/(N-1)d}{(cX+y)^2/4c} = \frac{4 + 4(\overline{y}-1)^2/(N-1)}{(\overline{y}+1)^2} = \frac{4}{N+3},$$

which is the profit ratio lower bound in (B.12), for the case $\overline{c} = 1$.

The case $\alpha_1 > 0$

We now consider the case where $\alpha_n > 0$ for every n. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\min_n\{\alpha_n\} = \alpha_1 = 1$.

Let $\mathbf{x}$ be a Cournot candidate in the original model with linear cost functions and an inverse demand function $p(\cdot)$. Suppose first that $x_1 = 0$. The second inequality in (B.1) implies that $p(X) \leq 1$. Lemma 1 implies that $X > 0$, so that there exists some n with $x_n > 0$. The first equality in (B.1) yields,

$$\alpha_n = p(X) + x_n p'(X) \leq 1.$$

Since $\alpha_n \geq 1$, we have that $p(X) = 1$ and $p'(X) = 0$. We observe that $\mathbf{x}$ satisfies the conditions in (4), and since $p(X) > 0$, we know that $\mathbf{x}$ maximizes the aggregate profit. However, since the cost functions are convex and $p(X) = \min_n\{\alpha_n\} = 1$, it is easy to see that the aggregate profit earned at $\mathbf{x}$ cannot be positive, a contradiction with Lemma 2. We therefore have $p(X) > 1$ and $x_1 > 0$.

As argued earlier, since $p(X) \notin \mathcal{P}$, we have $p'(X) < 0$. The first equality in (B.1), for $n = 1$, yields $y > 0$ and $x_1 = (y-1)/d$. We have

$$0 \leq \sum_{n=2}^{N} x_n = X - (y-1)/d, \tag{B.15}$$

from which we conclude that $X \geq (y-1)/d$.

From Proposition 4, the profit ratio of the vector $\mathbf{x}$ in the modified model, $\eta^0(\mathbf{x})$, cannot be more than its profit ratio in the original model, $\eta(\mathbf{x})$. Hence, to prove part (b), it suffices to show that $\eta^0(\mathbf{x}) \geq f^P(\bar{c}, N)$. For the modified model, the maximum aggregate profit is the optimal value of the following optimization problem,

$$\begin{aligned} &\text{maximize} \;\; qp^0(q) - q \\ &\text{subject to} \;\; q \geq 0. \end{aligned}$$

Since $dX \geq y-1$, the derivative of the aggregate profit at $q = X$ is nonpositive, and so the aggregate profit is nonincreasing with q on the interval $[X, \infty)$. Again, the aggregate profit is maximized in the interval $[0, X]$. Through a simple calculation we have:

(i) If $cX \geq y-1$, then the maximum aggregate profit is $(cX+y-1)^2/(4c)$, achieved at $Q_1 = (cX+y-1)/(2c)$.

(ii) If $cX \leq y-1$, then the maximum aggregate profit is $X(y-1)$, achieved at $Q_2 = X$.

Note that for $n = 1$ we have $\alpha_n x_n = x_n$. For $n \geq 2$, whenever $x_n > 0$, from the first equality in (B.1) we have $\alpha_n = y - x_n d$ and $\alpha_n x_n = (y - x_n d)x_n$. Since $\alpha_n \geq 1$, we have $y - 1 \geq x_n d$, for $n = 2, \ldots, N$. Therefore

$$(N-1)(y-1) \geq d \sum_{n=2}^{N} x_n = dX - (y-1),$$

which implies that

$$N(y-1) \geq dX. \tag{B.16}$$

Hence, in the modified model, the aggregate profit achieved at $\mathbf{x}$ is

$$\begin{aligned} X(y-1) - \sum_{n=1}^{N} \alpha_n x_n &= X(y-1) - (y-1)/d - \sum\nolimits_{n=2}^{N} (y - x_n d)x_n \\ &\geq X(y-1) - y\sum\nolimits_{n=2}^{N} x_n + \frac{(Xd-y-1)^2}{(N-1)d} \\ &= X(y-1) - (y-1)(X-(y-1)/d) + \frac{(Xd-y-1)^2}{(N-1)d} \\ &= \frac{(y-1)^2}{d} + \frac{(Xd-y-1)^2}{(N-1)d}, \end{aligned} \tag{B.17}$$

where the inequality is true because $\sum_{n=2}^{N} x_n^2$ is minimized when $x_2 = x_3 = \ldots = x_N$, subject to the constraint in (B.15). Note that c, d and $y-1$ are positive. If $y-1$ is replaced by y, then:

1. The aggregate profit achieved at $\mathbf{x}$, which is given in (B.17), is the same as that in (B.4).
2. The maximum aggregate profit is the same as that in the case where $\alpha_1 = 0$.
3. The constraints in (B.15) and (B.16) are equivalent to those in (B.2) and (B.3), respectively.

Let $\overline{y} = cX/(y-1)$ and $\overline{c} = c/d$. The desired results in (B.12) can be obtained by repeating the proof for the case $\alpha_1 = 0$.

Appendix C. Proof of Theorem 2

Without loss of generality, let supplier 1 have the largest market share, i.e., $x_1 = \max_n\{x_n\}$ and $r = x_1/X$. According to Proposition 1, the vector $\mathbf{x}$ remains a Cournot candidate in the modified model with linear cost functions, and Assumptions 1-5 still hold. Therefore, to lower bound the worst case profit ratio for Cournot candidates, we only need to derive a lower bound for the profit ratio of Cournot candidates for the case of linear cost functions. We therefore assume that $C_n(x_n) = \alpha_n x_n$ for each n. From the conditions (B.1), it is not hard to see that $\alpha_1 = \min_n\{\alpha_n\}$.

Since $p(X) \neq p(X^P)$, we have that $c > 0$. For conciseness, we let $y = p(X)$ throughout the proof. Through an approach similar to that used in the proof of Theorem 1, we will prove the theorem by considering separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$. According to Proposition 4, the profit ratio of the Cournot candidate $\mathbf{x}$ is lower bounded by the profit ratio $\eta^0(\mathbf{x})$ for the case of a piecewise linear inverse demand function of the form in (A.6).

The case $\alpha_1 = 0$

In part (b) of Theorem 1 we have shown that $p'(X) < 0$. The first equality in (B.1) yields $y > 0$, $x_1 = y/d$ and $r = y/dX$. For the case $cX \geq y$, in the proof of Theorem 1 (cf. Eq. (B.5)) we have shown that

$$\eta^0(\mathbf{x}) \geq \frac{y^2/d + (Xd-y)^2/((N-1)d)}{(cX+y)^2/4c}. \tag{C.1}$$

Note that c, d and y are positive. Substituting $r = y/dX$ and $\overline{c} = c/d$ to (C.1), we have

$$\eta^0(\mathbf{x}) \geq \frac{4\overline{c}^2 r^2 + \dfrac{4(\overline{c} - \overline{c}r)^2}{N-1}}{\overline{c}(\overline{c}+r)^2} \geq \frac{4\overline{c}^2 r^2}{\overline{c}(\overline{c}+r)^2} = \frac{4\overline{c}r^2}{(\overline{c}+r)^2}, \qquad 0 \leq r \leq \min\{\overline{c}, 1\}, \tag{C.2}$$

where the constraint $r \leq \overline{c}$ follows from $cX \geq y$.

For the case $cX \leq y$, we have $r \geq \overline{c}$. In the proof for Theorem 1 (cf. Eq. (B.8)) we have shown that

$$\eta^0(\mathbf{x}) \geq \frac{\dfrac{y^2}{d} + \dfrac{(Xd-y)^2}{(N-1)d}}{Xy} = \frac{\overline{c}^2 r^2 (N-1) + (\overline{c} - \overline{c}r)^2}{\overline{c}^2 r(N-1)} \geq r, \quad 0 < \overline{c} \leq r < 1. \tag{C.3}$$

Tightness

Given some $r \in (0,1)$, consider a model with $N \geq \lceil 1/r \rceil + 1$, and an affine inverse demand function $p^0(\cdot)$ of the form in (A.6), where $c/d = 1$ and $rdX = y$. The cost of supplier 1 is identically zero and[8]

$$C_n(x) = \left(y - \frac{d}{N-1}(X - rX)\right)x, \qquad n = 2, \ldots, N.$$

It is not hard to see that the vector with components

$$x_1 = rX, \qquad x_n = \frac{1}{N-1}(X - rX), \qquad n = 2, \ldots, N,$$

satisfies the conditions (5)-(6). It can be verified that $\mathbf{x}$ is a Cournot equilibrium. The maximum total profit is $(cX+y)^2/4c$, and is achieved at the monopoly output $\mathbf{x}^P = ((cX+y)/2c, 0, \ldots, 0)$. On the other hand, the aggregate profit achieved at $\mathbf{x}$ is given on the right-hand side of (B.4). We have

$$\eta^0(\mathbf{x}) = \frac{y^2/d + (Xd-y)^2/((N-1)d)}{(cX+y)^2/4c} = \frac{4r^2 + 4(1-r)^2/(N-1)}{(1+r)^2},$$

[8]Since $N > 1/r$ and $rdX = y$, we have that $C_n'(\cdot) > 0$.

and as the number of suppliers increases to infinity, the profit ratio of the Cournot equilibrium converges to $4r^2/(1+r)^2$.

The case $\alpha_1 > 0$

The proof is similar to the case that $\alpha_1 = 0$ and is omitted.

Appendix D. Proof of Corollary 3

Let $\mathbf{x}$ be a Cournot candidate. For convex inverse demand functions, we have shown in the proof of Corollary 2 that $p'(X)$ must exist. If $p(X) \in \mathcal{P}$, Proposition 3 shows that $\eta(\mathbf{x}) = 1$, and the desired results trivially hold. Now consider the case $p(X) \notin \mathcal{P}$. Let $\mathbf{x}^P$ maximize the aggregate profit. We first show that the aggregate supply at a Cournot candidate is at most s, and then argue that the aggregate supply at a monopoly output is at least t. The desired results will follow from the fact that both the functions, $f^P(\cdot, N)$ and $g^P(\cdot, r)$, are nonincreasing.

Step 1: The aggregate supply at the Cournot candidate, X, is no more than s.

We first show that $X \le s$. Since $p'(X)$ exists, we know that $\mathbf{x}$ satisfies the necessary conditions in (B.1). Lemma 1 shows that $X > 0$. For a supplier n with $x_n > 0$, the first equality in (B.1) implies that

$$p(X) \ge C_n'(x_n) \ge C_n'(0) \ge \min_n\{C_n'(0)\},$$

where the first inequality is true because $p(\cdot)$ is nonincreasing and $p'(X) \le 0$, and the second inequality follows from the convexity of $C_n(\cdot)$.

For any Cournot candidate $\mathbf{x}$ with $p(X) > 0$, we now argue that $p(X) > \min_n\{C_n'(0)\}$. Suppose not. We have $p(X) = \min_n\{C_n'(0)\}$, $p(X) = C_n'(x_n)$, and $p'(X) = 0$. We observe that $\mathbf{x}$ satisfies the conditions in (4); since $p(X) > 0$, we know that $\mathbf{x}$ maximizes the aggregate profit. However, since cost functions are convex and $p(X) = \min_n\{C_n'(0)\}$, it is easy to see that the aggregate profit earned at $\mathbf{x}$ cannot be positive, a contradiction with Lemma 2. Since $p(X) > \min_n\{C_n'(0)\}$ and $p(\cdot)$ is nonincreasing, we conclude that $X \le s$.

Step 2: The aggregate supply at the monopoly output, X^P, is at least t.

Lemma 2 implies that $X^P > 0$. Applying conditions (4) to some n with $x_n^P > 0$, we know that $p'(X^P)$ exists, because $\partial_- p\left(X^P\right) \leq \partial_+ p\left(X^P\right)$. The conditions (4) imply that

$$C_n'(x_n^P) \geq p(X^P) + X^P p'(X^P), \qquad \forall n. \tag{D.1}$$

Since $X^P \geq x_n^P$ and the cost functions are convex, we have

$$C_n'(X^P) \geq p(X^P) + X^P p'(X^P), \qquad \forall n,$$

which implies that $X^P \geq t$. Proposition 2 shows that $X^P < X$, and therefore we have $t \leq X^P < X \leq s$. Since $\partial_- p(s) < 0$, and $p(\cdot)$ is convex and nonincreasing, we have

$$\overline{c} = c/d \leq \partial_+ p(t)/\partial_- p(s).$$

The desired result follows from Theorems 1 and 2, as well as the fact that both the functions, $f^P(\cdot, N)$ and $g^P(\cdot, r)$, are nonincreasing.