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NP-HARDNESS OF SOME LINEAR CONTROL DESIGN PROBLEMS*

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Abstract. We show that some basic linear control design problems are NP-hard, implying that, unless P=NP, they cannot be solved by polynomial time algorithms. The problems that we consider include simultaneous stabilization by output feedback, stabilization by state or output feedback in the presence of bounds on the elements of the gain matrix, and decentralized control. These results are obtained by first showing that checking the existence of a stable matrix in an interval family of matrices is NP-hard.

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1. Introduction. Consider the following three problems; the first was mentioned as a "major open problem in systems and control theory" in a recent survey [5] of experts in the systems and control field, and the other two were mentioned indirectly.

Stabilization by static output feedback. This is perhaps the most basic problem in control theory. We are given a linear system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned}$$

and we consider a static feedback control law of the form

$$u(t) = Ky(t).$$

The resulting closed loop system is

$$\dot{x}(t) = (A + BKC)x(t).$$

The problem is to find necessary and sufficient conditions on the triplet of real matrices (A, B, C) under which there exists a feedback gain matrix K such that $A + BKC$ is stable. In the case of state feedback ($C = I$), a necessary and sufficient stabilizability condition is given by the stabilizability of the pair (A, B) [17]. However, if C is not invertible, no general necessary and sufficient conditions are known.

Simultaneous stabilization by static output or state feedback. (This problem should not be confused with what is usually referred to as the "simultaneous stabilization problem" [16, 4], in which dynamic—instead of static—compensation is sought.) Our second problem is a generalization of the static output feedback problem. Suppose

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that for each $i = 1, \dots, k$ we are given a linear system

$$\dot{x}(t) = A_i x(t) + B_i u(t),$$
$$y(t) = C_i x(t).$$

Under the feedback control law,

$$u(t) = Ky(t),$$

the ith closed loop system is

$$\dot{x}(t) = (A_i + B_i K C_i) x(t).$$

The problem is to find conditions on the triplets of real matrices (A_i, B_i, C_i), $i = 1, \dots, k$, under which there exists a matrix K such that $A_i + B_i K C_i$ is stable for each i. This problem is unsolved, even if $C_i = I$ for all i (simultaneous stabilization by state feedback).

Stabilization by decentralized static output feedback. We now impose some structure on the feedback gains. Consider a linear system of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{k} B_i u_i(t),$$
$$y_i(t) = C_i x(t), \qquad i = 1, \dots, k,$$

and suppose that we are interested in a static decentralized controller of the form

$$u_i(t) = K_i y_i(t), \qquad i = 1, \dots, k.$$

The closed loop system is

$$\dot{x}(t) = \left(A + \sum_{i=1}^{k} B_i K_i C_i \right) x(t),$$

which is of the same form as in stabilization by static output feedback, except that several of the entries of K are forced to zero. This leads us to the problem of finding conditions on the triplet of real matrices (A, B, C) under which there exists a matrix K with a given structure such that $A + BKC$ is stable. The problem can be further constrained by requiring the matrix structure to be block diagonal, the blocks to have a bounded norm, or the blocks to be identical (we discuss all of these cases later).

The reader is referred to [3, p. 420], where the above three problems are presented and motivated and where references can be found. A common feature of these three problems is that, although they are easy to state, neither closed form nor efficient algorithmic solutions are known. It is rather improbable that closed form solutions to these problems are possible. On the other hand, algorithmic solutions do exist, as we now argue.

All of the problems that we have described are finitely parametrized. They all involve the search for a controller—the (possibly partitioned) matrix K—which can be specified in terms of finitely many real parameters. In theory, it is thus possible to apply the following methodology: (a) parametrize the gain matrix K in terms of finitely many real coefficients; (b) express the matrix stability condition(s) in terms of the coefficients of the system(s) and of the controller; (c) use the Routh–Hurwitz test on the resulting characteristic polynomial(s). One is then left with a (large) set of multivariable polynomial inequalities that have to be simultaneously satisfied for some choice of the controller coefficients. As explained in [1], checking the existence of controller coefficients that satisfy this system of multivariable inequalities can be performed using the Tarski–Seidenberg elimination theory. The Tarski–Seidenberg elimination method leads, after a finite number of rational operations, to a yes-no answer regarding the existence of a solution. The method is systematic and amenable to computer implementation. Thus, *all three problems described above are algorithmically solvable.*

The advantage of the Tarski–Seidenberg method is its generality; its drawback is the fact that its computational complexity increases at least exponentially. The examples that can be worked on paper are very small (the example given in [1] involves only two parameters), and computer algorithms cannot digest more than five or six parameters in reasonable time.

In this paper we show that some of the above problems and their variations are very unlikely to allow for *efficient* algorithmic solutions. We adhere to the general consensus in computer science that identifies algorithmic efficiency with polynomial time computability. We then show that some of the above problems are NP-hard [8, 13], meaning that every problem in NP can be reduced to them. Thus, unless P=NP, these problems are not polynomial time solvable.

Our results are as follows (see later for precise definitions):

1. The static output feedback stabilization problem is NP-hard if one constrains the coefficients of the controller K to lie in prespecified intervals. The same is true in the case of static *state* feedback ($C = I$). We have not been able to establish the complexity of the problem in the absence of constraints on K, but we conjecture that it is also NP-hard.
2. Simultaneous stabilization by output feedback is NP-hard.
3. Stabilization by decentralized static output feedback is NP-hard if one imposes a bound on the norm of the controller or if the blocks are constrained to be identical.

These results will be proved as corollaries of the following main theorem: testing for the presence of a stable matrix in a family of matrices whose members have entries that are either fixed to some given real number or vary in the closed unit interval $[-1, 1]$ is an NP-hard problem. This latter result complements a recent theorem of Nemirovskii [11], who showed that testing for the stability of *all* elements of such a family of matrices is an NP-hard problem. Our proof is in fact inspired from his. This general research direction was initiated by Poljak and Rohn, who showed that checking nonsingularity of an interval family of matrices is NP-hard [14]. In other related research, NP-hardness of the computation of the structured singular value μ was shown by Braatz et al. [6] for the case where some perturbations are complex. (NP-completeness for the case of real perturbations was a corollary of the results of Poljak and Rohn.) Also, Coxson and DeMarco show that approximating the minimal perturbation scaling to achieve instability in an interval matrix is MAX-SNP-hard [7]. See also [15] for a review of other complexity results for problems in control theory.

In the next section, we prove the main result and derive some general corollaries. In the last section we link these results with the linear control design problems mentioned in this introduction.

2. Checking the existence of a stable matrix in an interval family of matrices is NP-hard. In this section we show that checking the existence of a stable matrix in a unit interval family of matrices is an NP-hard problem (a unit interval family of matrices is a family of matrices whose members have entries that are either fixed to some given real number or vary in the closed unit interval $[-1,1]$). We prove this result by means of a polynomial time reduction from the following problem, which is already known to be NP-complete [10, 8].

PARTITION

Instance: A positive integer l, a set of l integers $a_i \in \mathcal{Z}$.

Question: Do there exist $t_1, \ldots, t_l \in \{-1,+1\}$ such that $\sum_{i=1}^{l} a_i t_i = 0$?

We now formally define the problem of interest.

STABLE MATRIX IN UNIT INTERVAL FAMILY

Instance: A positive integer n, a partition of $I = \{(i,j) : 1 \le i,j \le n\}$ into disjoint sets I_1 and I_2, rational numbers a_{ij}^* for $(i,j) \in I_1$.

Question: Does the set $\mathcal{A}$ of $n \times n$ matrices defined by

$$\mathcal{A} = \{A = (a_{ij}) : a_{ij} = a_{ij}^* \text{ for } (i,j) \in I_1,\ a_{ij} \in [-1,1] \text{ for } (i,j) \in I_2\}$$

contain a stable matrix?

Remark. Throughout this paper, when writing "stable" we actually mean "asymptotically stable," i.e., "all eigenvalues have a negative real part." A slightly different problem is obtained if we are interested in marginal stability ("all eigenvalues have a nonpositive real part"). We call this second problem MARGINALLY STABLE MATRIX IN UNIT INTERVAL FAMILY.

The main result of this paper is as follows.

THEOREM 1. STABLE MATRIX IN UNIT INTERVAL FAMILY *and* MARGINALLY STABLE MATRIX IN UNIT INTERVAL FAMILY *are NP-hard.*

Proof. We prove NP-hardness of STABLE MATRIX IN UNIT INTERVAL FAMILY. NP-hardness of MARGINALLY STABLE MATRIX IN UNIT INTERVAL FAMILY can be shown in a similar way; we make a comment on this at the end of the proof.

Since PARTITION is NP-complete, it suffices to show that any instance of PARTITION can be transformed in polynomial time into an equivalent instance of STABLE MATRIX IN UNIT INTERVAL FAMILY.

Let $a_i \in \mathcal{Z}$ $(i = 1, \ldots, l)$ be an instance of PARTITION. We construct a unit interval matrix as follows. Let m be a positive integer such that $l < m = k^2$ for some positive integer k, and define the m-dimensional vector a by $a^T = (a_1, a_2, \ldots, a_l, 0, \ldots, 0) \in \mathcal{Z}^m$ (the superscript ^{T} denotes matrix transposition). Let $\gamma = a^T a$, $\beta = 1 - 1/(2m(1+\gamma))$, and

$$A(x,y) = \begin{pmatrix} -k(I_m + aa^T) & y \\ x^T & k\beta \end{pmatrix}, \tag{1}$$

with I_m the identity matrix of size m and $x, y \in \Re^m$ (note that $\gamma > 0$ and $0 < \beta < 1$).

The set of matrices

$$\mathcal{A} = \{A(x,y) : x, y \in [-1,1]^m\} \tag{2}$$

forms an instance of STABLE MATRIX IN UNIT INTERVAL FAMILY and is constructed in polynomial time from the initial instance of PARTITION. It remains thus to show that $\mathcal{A}$ contains a stable matrix if and only if there exist $t_i \in \{-1,+1\}$ such that $\sum_{i=1}^{l} a_i t_i = 0$. We prove this in two steps.

Assume first that $t_i \in \{-1, +1\}$ satisfy $\sum_{i=1}^{l} a_i t_i = 0$. Define $x_0^T = (t_1, t_2, \ldots, t_l, 1, \ldots, 1) \in \mathcal{Z}^m$, $y_0 = -x_0$, and note that $a^T x_0 = x_0^T a = 0$. We claim that the matrix $A_0 = A(x_0, y_0) \in \mathcal{A}$ is stable. Indeed, A_0 can be decomposed as

$$A_0 = A_1 + A_2 + A_3 \tag{3}$$

$$= -kI_{m+1} + \begin{pmatrix} -kaa^T & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -x_0 \\ x_0^T & k(1+\beta) \end{pmatrix}. \tag{4}$$

The spectrum of A_0 is the spectrum of $A_2 + A_3$ shifted to the left by k. The matrix A_0 will thus be stable, provided that the real part of every eigenvalue of $A_2 + A_3$ is strictly less than k.

The matrix A_2 has rank one; it has one eigenvalue at $-k\gamma$ and m eigenvalues at the origin. The characteristic polynomial of the matrix A_3 is

$$s^{m-1}(s^2 - k(1+\beta)s + k^2), \tag{5}$$

whose roots are either at the origin or have a real part equal to $k(1+\beta)/2$ which is always strictly less than k, since we have already observed that $0 < \beta < 1$.

Due to the fact that $a^T x_0 = x_0^T a = 0$, we have $A_2 A_3 = A_3 A_2 = 0$. Let λ and w be an eigenvalue and an eigenvector, respectively, of $A_2 + A_3$. Thus, $(A_2 + A_3)w = \lambda w$. Multiplying by A_2, we obtain $A_2^2 w = \lambda A_2 w$. If $A_2 w \neq 0$, then λ is an eigenvalue of A_2. If $A_2 w = 0$, then λ is an eigenvalue of A_3. Consequently, every eigenvalue of $A_2 + A_3$ is either an eigenvalue of A_2 or of A_3. These eigenvalues have a real part which is smaller than k, and by our earlier comment, the matrix $A_0 \in \mathcal{A}$ is stable.

For the reverse implication, assume that $\mathcal{A}$ contains a stable matrix and let $x_0, y_0 \in [-1, 1]^m$ be such that $A_0 = A(x_0, y_0) \in \mathcal{A}$ is stable. Consider then the parametrized family of matrices

$$B(\theta) = A(\theta x_0, \theta y_0)/k. \tag{6}$$

We now study the dependence of the stability of $B(\theta)$ on the variable $\theta \in [0, 1]$. When $\theta = 0$, we have

$$B(0) = \begin{pmatrix} -(I_m + aa^T) & 0 \\ 0 & \beta \end{pmatrix}. \tag{7}$$

The matrix $-(I_m + aa^T)$ is negative definite, hence stable, and thus $B(0)$ has a single unstable eigenvalue (at $\beta > 0$). When $\theta = 1$, we have $B(1) = A_0/k$, and so $B(1)$ is stable since A_0 is.

The eigenvalues of $B(\theta)$ are symmetric with respect to the real axis (complex conjugate), and they vary continuously with θ. When moving from $\theta = 0$ to $\theta = 1$, we move from a configuration where there is exactly one unstable eigenvalue to a configuration with no unstable eigenvalues. When a conjugate pair of eigenvalues crosses the $j\omega$ axis, the number of unstable eigenvalues changes by an even number. Thus, for the number of unstable eigenvalues to change from one to zero, some eigenvalue must cross the $j\omega$ axis at the origin. Therefore, there exists some $\theta_0 \in (0, 1)$ for which $B(\theta_0)$ has an eigenvalue at the origin and $B(\theta_0)$ is singular. Elementary matrix manipulations show that the singularity condition for $B(\theta_0)$ is equivalent to

$$\theta_0^2 x_0^T (I_m + aa^T)^{-1} y_0 = -k^2 \beta. \tag{8}$$

A standard inversion formula [9, p. 19] gives

$$\theta_0^2 x_0^T (I_m - aa^T/(1+\gamma)) z_0 = k^2 \beta, \tag{9}$$

where we have defined $z_0 = -y_0$ and used the definition $\gamma = a^T a$. Remembering that $m = k^2$ and $\theta_0 \in (0,1)$, we finally obtain

$$x_0^T(I_m - aa^T/(1+\gamma))z_0 > m\beta. \tag{10}$$

The matrix $(I_m - aa^T/(1+\gamma))$ is symmetric and positive definite. Using also the fact that the maximum of a convex function over a bounded polyhedron is attained at an extreme point, we obtain

$$\max_{x,y\in[-1,1]^m} x^T(I_m - aa^T/(1+\gamma))y = \max_{x\in[-1,1]^m} x^T(I_m - aa^T/(1+\gamma))x \tag{11}$$

$$= \max_{x\in\{-1,1\}^m} x^T(I_m - aa^T/(1+\gamma))x \tag{12}$$

$$= m - \min_{x\in\{-1,1\}^m} (x^T a)^2/(1+\gamma). \tag{13}$$

In particular, this shows that

$$m - \min_{x\in\{-1,1\}^m} (x^T a)^2/(1+\gamma) \geq x_0^T(I - aa^T/(1+\gamma))z_0. \tag{14}$$

Combining inequalities (10) and (14), we obtain

$$m - \min_{x\in\{-1,1\}^m} (x^T a)^2/(1+\gamma) > m\beta. \tag{15}$$

Using the definition of β, we finally arrive at

$$\min_{x\in\{-1,1\}^m} (x^T a)^2 < 1/2. \tag{16}$$

The left-hand side in this inequality is a nonnegative integer; we are thus forced to the conclusion

$$\min_{x\in\{-1,1\}^m} (x^T a)^2 = 0. \tag{17}$$

Assume that the minimum in (17) is obtained for $x^T = (x_1, x_2, \ldots, x_l, \ldots, x_m)$; we conclude the proof by setting $t_i = x_i$ for $i = 1, \ldots, l$.

Let us now briefly comment on the case where we are interested in marginal stability. NP-hardness for this case can be obtained by a small adaptation of the preceding proof. Let, as before, $a_i \in \mathcal{Z}$ $(i = 1, \ldots, l)$ be an instance of PARTITION. We construct an interval matrix as follows. Let m be a positive integer such that $l < m = k^2$ for some positive integer k and define the m-dimensional vector a by $a^T = (a_1, a_2, \ldots, a_l, 0, \ldots, 0) \in \mathcal{Z}^m$ and

$$A(x,y) = \begin{pmatrix} -k(I_m + aa^T) & y \\ x^T & k \end{pmatrix}. \tag{18}$$

The set of matrices $\mathcal{A} = \{A(x,y) : x, y \in [-1,1]^m\}$ forms an instance of MARGINALLY STABLE MATRIX IN UNIT INTERVAL FAMILY and is constructed in polynomial time from the initial instance of PARTITION. Moreover, by the same argument as above, it is clear that $\mathcal{A}$ contains a marginally stable matrix if and only if there exist $t_i \in \{-1, +1\}$ such that $\sum_{i=1}^{l} a_i t_i = 0$. This shows the equivalence between the instances and hence proves the second part of the theorem. $\square$

Suppose now that we change the problem by including the additional requirement that the matrix A must be symmetric. Consider the problem of minimizing λ subject to $\lambda I - A$ being a positive semidefinite symmetric matrix and subject to the interval constraints on A. This is a semidefinite programming problem and can be solved, within any desired accuracy ϵ, in time which is polynomial in the size of the problem and the "size" $\log(1/\epsilon)$ of ϵ. Furthermore, the optimal cost in this minimization problem is less than or equal to zero (respectively, negative) if and only if there exists a marginally stable (respectively, stable) matrix A in the family. This argument, brought to our attention by M. Overton [12], comes close but does not quite establish polynomiality of the problem STABLE MATRIX IN UNIT INTERVAL FAMILY for the symmetric case; that would require an exact (as opposed to approximate) polynomial time solution of the semidefinite programming problem. If the symmetric problem is indeed polynomial time solvable, this would be in contrast to the results of Nemirovskii [11], who showed that deciding the stability of all elements of the interval family is NP-hard even if one restricts to symmetric matrices.

As a direct application of our main theorem, we introduce a few matrix and polynomial stability problems and show that they are NP-hard.

STABLE MATRIX IN INTERVAL FAMILY

Instance: A positive integer n, rational numbers $\underline{a}_{ij}, \overline{a}_{ij}$ for $1 \leq i, j \leq n$.

Question: Does there exist a stable matrix $A = (a_{ij})$ with $\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}$?

STABLE MATRIX IN RANK ONE PERTURBED MATRIX

Instance: Positive integers n, k, and $k+1$ real $n \times n$ matrices $A_0, A_1, \ldots, A_k$ with rational entries, all of which have rank one, with the exception of A_0.

Question: Do there exist real values $q_i^* \in [-1, 1]$ such that $A = A_0 + q_1^* A_1 + \cdots + q_k^* A_k$ is stable?

STABLE POLYNOMIAL IN FAMILY OF BILINEAR POLYNOMIALS

Instance: A positive integer r, a multivariable polynomial $p(x, q_1, \ldots, q_r)$ with rational coefficients whose dependence on the real variables q_i is bilinear.

Question: Do there exist real values $q_i^* \in [-1, 1]$ for which the polynomial $p(x, q_1^*, \ldots, q_r^*)$ is stable?

COROLLARY 1. *The above three problems are all NP-hard.*

Proof. STABLE MATRIX IN INTERVAL FAMILY is NP-hard because it is a generalization of STABLE MATRIX IN UNIT INTERVAL FAMILY.

A matrix A in the unit interval family defined by I_1 and a_{ij}^*, $(i,j) \in I_1$, can be written in the form

$$A = A_0 + \sum_{(i,j) \notin I_1} q_{ij} A_{ij},$$

where A_0 has entries

$$\begin{aligned} a_{ij}^0 &= a_{ij}^* \quad &&\text{if } (i,j) \in I_1, \\ &= 0 &&\text{if } (i,j) \notin I_1. \end{aligned}$$

A_{ij} is a matrix with all entries equal to zero except for the (i,j)th entry, which is equal to 1, and $q_{ij} \in [-1, 1]$; note that A_{ij} has rank one. This reduces STABLE MATRIX IN UNIT INTERVAL FAMILY to STABLE MATRIX IN RANK ONE PERTURBED MATRIX and shows that the latter problem is NP-hard.

In order to prove that STABLE POLYNOMIAL IN FAMILY OF BILINEAR POLYNOMIALS is NP-hard, we argue as in the proof of Theorem 1. Let $a_i \in \mathcal{Z}$ $(i = 1, \ldots, l)$ be an instance of PARTITION. Let m be a positive integer such that $l < m = k^2$ for some positive integer k and define $\beta = 1 - 1/(2m(1 + \sum_{i=1}^{l} a_i^2))$ and

$$A(q_1, \ldots, q_k, q_{k+1}, \ldots, q_{2k}) = \begin{pmatrix} -k(I_m + aa^T) & (q_{k+1}, \ldots, q_{2k})^T \\ (q_1, \ldots, q_k) & k\beta \end{pmatrix}.$$

From the proof of Theorem 1, we know that the set of matrices $\mathcal{A} = \{A(q_1, \ldots, q_k, q_{k+1}, \ldots, q_{2k}) : q_i \in [-1, 1]\}$ contains a stable matrix if and only if there exist $t_i \in \{-1, +1\}$ such that $\sum_{i=1}^{l} a_i t_i = 0$. The set of matrices $\mathcal{A}$ contains a stable matrix if and only if the multivariable polynomial $p(x, q_1, \ldots, q_{2k}) = \det(xI_{2k} - A(q_1, \ldots, q_k, q_{k+1}, \ldots, q_{2k}))$ is stable for some choice of $q_i \in [-1, 1]$. The latter polynomial is bilinear in the variables q_i. We therefore have an instance of STABLE POLYNOMIAL IN FAMILY OF BILINEAR POLYNOMIALS which is equivalent to the original instance of PARTITION. ◻

Remarks.

1. All three problems addressed by Corollary 1 remain NP-hard if “stability” is replaced by “marginal stability”; the proof is similar.

2. By a similar proof, both Theorem 1 and Corollary 1 remain valid if the interval constraints $a_{ij} \in [-1, 1]$ are replaced by the open interval constraints $a_{ij} \in (-1, 1)$.

3. The decision problem for the existential theory of the reals is solvable in $s^{k+1} d^{O(k)}$ arithmetic operations where k denotes the number of variables, s is the number of polynomial (in)equalities, and d is the highest polynomial degree [2]. This shows that for fixed k, a polynomial time algorithm is possible. In particular, STABLE MATRIX IN INTERVAL FAMILY becomes polynomial time solvable if an a priori bound is given on the size of the matrix. The problems discussed in Corollary 1 also become polynomial time solvable when suitably constrained.

3. Application to linear control design problems. As explained in the introduction, our initial motivation for this work was to address the computational complexity of linear control design problems. We now introduce some such problems and show that they are NP-hard.

STATE FEEDBACK STABILIZATION BY BOUNDED CONTROLLER

Instance: A positive integer n, $n \times n$ matrices A and B with rational coefficients, rational numbers $\underline{k}_{ij}, \overline{k}_{ij}$ for $1 \le i, j \le n$.

Question: Does there exist a real matrix $K = (k_{ij})$ satisfying $\underline{k}_{ij} \le k_{ij} \le \overline{k}_{ij}$ and such that $A + BK$ is stable?

SIMULTANEOUS STABILIZATION BY OUTPUT FEEDBACK

Instance: Positive integers n, m, p, k, a collection of k triplets of matrices (A_i, B_i, C_i) with rational coefficients of respective sizes $n \times n, n \times m, p \times n$.

Question: Does there exist a real $m \times p$ matrix K such that $A_i + B_i K C_i$ is stable for all $i = 1, \ldots, k$?

DECENTRALIZED OUTPUT FEEDBACK STABILIZATION BY NORM BOUNDED CONTROLLER

Instance: Positive integers n and k with $n \ge k$, $n \times n$ matrices A, B and C with rational coefficients. A partition of n into k positive integers $n = n_1 + n_2 + \cdots + n_k$.

Question: Does there exist a $n \times n$ block-diagonal matrix K with blocks K_i of successive sizes $n_i \times n_i$ and $||K_i|| < 1$ such that $A + BKC$ is stable?

DECENTRALIZED STABILIZATION WITH IDENTICAL CONTROLLERS

Instance: Positive integers n_1, n_2, three $(n_1 n_2) \times (n_1 n_2)$ matrices A, B and C with rational coefficients.

Question: Does there exist a $n_1 \times n_1$ matrix M such that the $(n_1 n_2 \times n_1 n_2)$ block diagonal matrix K constructed with n_2 identical blocks M is such that $A + BKC$ is stable?

COROLLARY 2. *The above four problems are all NP-hard.*

Proof. (a) STATE FEEDBACK STABILIZATION BY BOUNDED CONTROLLER: Let n and $\underline{a}_{ij}$, $\overline{a}_{ij}$, for $1 \leq i,j \leq n$ be an instance of STABLE MATRIX IN INTERVAL FAMILY. An equivalent instance of STATE FEEDBACK STABILIZATION BY BOUNDED CONTROLLER is given by n, $A = 0$, $B = I_n$, $\underline{k}_{ij} = \underline{a}_{ij}$, and $\overline{k}_{ij} = \overline{a}_{ij}$ for $1 \leq i,j \leq n$.

(b) SIMULTANEOUS STABILIZATION BY OUTPUT FEEDBACK:

We prove NP-hardness for the case of marginal stability. Let n and $\underline{a}_{kl}, \overline{a}_{kl}$ ($1 \leq i,j \leq n$) be an instance of STABLE MATRIX IN INTERVAL FAMILY. Define the $n \times n$ matrices A^+_{ij}, A^-_{ij}, B_i, and C_j by

$A^+_{ij} = (a_{kl})$ with

$$\begin{aligned} a_{kl} &= -\overline{a}_{ij} \text{ if } (k,l) = (1,1), \\ &= 0 \text{ otherwise;} \end{aligned}$$

$A^-_{ij} = (a_{kl})$ with

$$\begin{aligned} a_{kl} &= \underline{a}_{ij} \text{ if } (k,l) = (1,1), \\ &= 0 \text{ otherwise;} \end{aligned}$$

$B_i = (b_{kl})$ with

$$\begin{aligned} b_{kl} &= 1 \text{ if } (k,l) = (1,i), \\ &= 0 \text{ otherwise;} \end{aligned}$$

and $C_j = (c_{kl})$ with

$$\begin{aligned} c_{kl} &= 1 \text{ if } (k,l) = (j,1), \\ &= 0 \text{ otherwise.} \end{aligned}$$

It is immediate to see that $(A^+_{ij} + B_i K C_j)$ is marginally stable if and only if $k_{ij} \leq \overline{a}_{ij}$, and similarly, $(A^-_{ij} + B_i K C_j)$ is marginally stable if and only if $k_{ij} \geq \underline{a}_{ij}$. Thus, if we require the simultaneous stabilization of the $2n^2 + 1$ triplets $(0, I, I), (A^+_{ij}, B_i, C_j)$, and (A^-_{ij}, B_i, C_j) for $1 \leq i,j \leq n$, we have constructed an equivalent instance of SIMULTANEOUS STABILIZATION BY OUTPUT FEEDBACK.

(c) DECENTRALIZED OUTPUT FEEDBACK STABILIZATION BY NORM BOUNDED CONTROLLER: We prove that the problem is NP-hard even for the special case where all blocks are of size 1×1, in which case $A + BKC$ can be written as $A + \sum_{i=1}^n k_i b_i c_i^T$, where b_i is the ith column of B, c_i^T is the ith row of C, and k_i is the ith diagonal entry of K. Given that an arbitrary rank one matrix can be expressed in the form bc^T for some vectors b and c, it follows that every instance of STABLE MATRIX IN RANK ONE PERTURBED MATRIX can be expressed as an instance of DECENTRALIZED OUTPUT FEEDBACK STABILIZATION BY NORM BOUNDED CONTROLLER

(d) DECENTRALIZED STABILIZATION WITH IDENTICAL CONTROLLERS: We prove NP-hardness for the case of marginal stability. Consider k triplets of $n \times n$ matrices (A_i, B_i, C_i) that form an instance of SIMULTANEOUS STABILIZATION BY OUTPUT FEEDBACK. We define an equivalent instance of DECENTRALIZED STABILIZATION WITH IDENTICAL CONTROLLERS by letting $n_1 = n, n_2 = k, A = A_1 \oplus A_2 \oplus \cdots \oplus A_k, B = B_1 \oplus B_2 \oplus \cdots \oplus B_k$ and $C = C_1 \oplus C_2 \oplus \cdots \oplus C_k$, where $\oplus$ denotes direct sum of matrices. ∎

Remarks.

1. For some of the problems, we provided the proof for the case of stability; for others, we dealt with marginal stability. With little work and using the remarks at the end of the preceding section, it is easily shown that all problems are NP-hard for the case of either stability or marginal stability.

2. STATE FEEDBACK STABILIZATION BY BOUNDED CONTROLLER is easily shown to remain NP-hard even if the bounds $\underline{k}_{ij}, \overline{k}_{ij}$ are constrained to be either 0 or 1. We have assumed that we are dealing with square systems; the more general case of rectangular systems is at least as hard and is therefore also NP-hard. Finally, the problem of *output* feedback stabilization by a bounded controller is at least as hard as that of *state* feedback and is thus also NP-hard.

3. Our proof shows that SIMULTANEOUS STABILIZATION BY OUTPUT FEEDBACK remains NP-hard even if all the matrices involved are of the same size ($n = m = p$). The degenerate case $m = p = 1$ corresponds to simultaneous stabilization of single-input, single-output systems by proportional feedback and can be solved in polynomial time. (An argument for this follows from footnote 1 on p. 54 of [1].) For a priori fixed n, m, and p, the problem can also be solved in polynomial time (see Remark 3 in section 2). We do not know whether the state feedback formulation of this problem is NP-hard.

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